1.1 An introduction to limits

Definition

(Informal definition)

If we can make the values of a function $f(x)$ get arbitrarily close to a number L by taking x-values that are sufficiently close to c (but not equal to c), then we say the *limit* of $f(x)$ as x approaches c is L, and we write

$$\lim_{x \to c} f(x) = L$$

Remark

We need to define more clearly what is means to be arbitrarily close and sufficiently close in order to make the informal definition of limit given above more rigorous. We'll do that in the next section. For now, we'll use this informal definition to explore the idea of a limit by estimating limits (a) graphically and (b) numerically.

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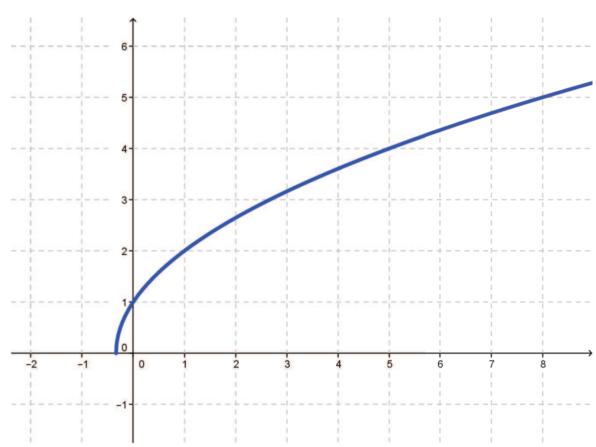
Estimating limits graphically

Example

Find

- $\lim_{x \to 5} \sqrt{3x+1} = $ ____________
- $\lim_{x \to 1} \sqrt{3x+1} = $ ____________
- $\lim_{x \to 0} \sqrt{3x+1} = $ ____________
- $\lim_{x \to 8} \sqrt{3x+1} = $ ____________
- $\lim_{x \to 2} \sqrt{3x+1} = $ ____________



$y = \sqrt{3x+1}$

Estimating limits numerically

Example

Find $\lim_{x\to 3} \dfrac{2x^2 - 2x - 12}{x - 3}$

Solution

Pick x-values near 3 (but not equal to 3) and compute the value of the function.

x	$\frac{2x^2-2x-12}{x-3}$	x	$\frac{2x^2-2x-12}{x-3}$
2.9		3.1	
2.99		3.01	
2.999		3.001	

How close to 3 would x have to be in order to get within

- 0.0002 of the limit?
- 0.1 of the limit?

When limits do not exist

Remark

A function may not have a limit for all values of x. There are three ways a limit may fail to exist as x approaches c:

1. The function may approach different values on either side of c or fail to exist near c.
2. The function may grow without bound (i.e. tend toward $+\infty$ or $-\infty$) as x approaches c.
3. The function may oscillate infinitely often as x approaches c without staying near any single value.

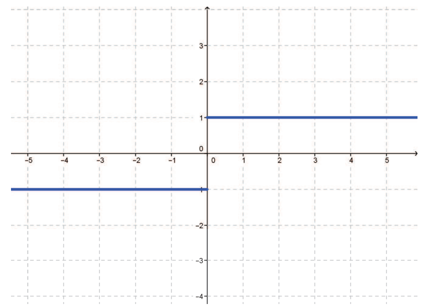
In the following slides we give examples of each case.

When limits do not exist: (1) different values from left and right

Example

Consider

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & x > 0 \\ \frac{-x}{x} = -1 & x < 0 \end{cases}$$



$y = |x|/x$

$\lim_{x \to 0} \frac{|x|}{x}$ does not exist because $\lim_{x \to 0^+} \frac{|x|}{x} = 1$ (the right-hand limit) while $\lim_{x \to 0^-} \frac{|x|}{x} = -1$ (the left-hand limit).

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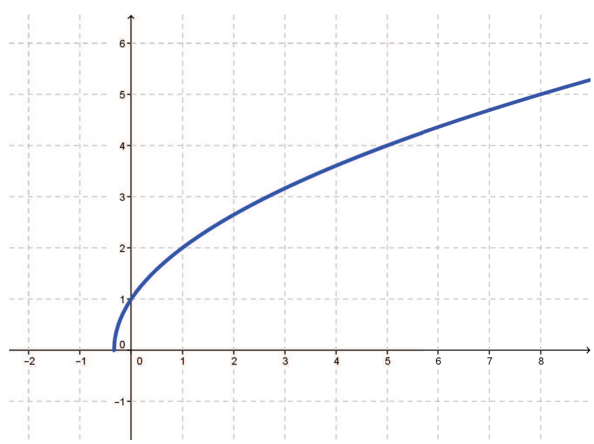
When limits do not exist: (1) function fails to exist near c

Example

Consider

$$f(x) = \sqrt{3x + 1}$$



$y = \sqrt{3x + 1}$

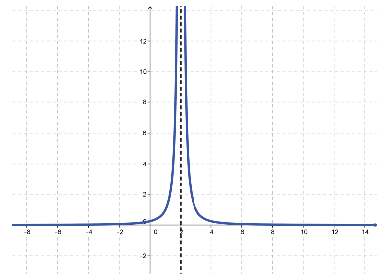
$\lim_{x \to -1/3} \sqrt{3x + 1}$ does not exist because $f(x) = \sqrt{3x + 1}$ does not exist for $x < -1/3$.

When limits do not exist: (2) function grows without bound

Example

Consider

$$f(x) = \frac{1}{(x-2)^2}$$



$y = 1/(x-2)^2$

$\lim_{x\to 2} 1/(x-2)^2$ does not exist because there is no number L that the function $1/(x-2)^2$ gets closer to as x approaches 2; indeed, the values of the function $1/(x-2)^2$ can be made as large as we'd like by taking x sufficiently close to 2. In other words, the function grows without bound as x approaches 2.

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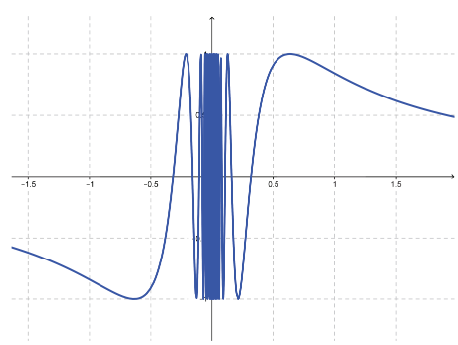
When limits do not exist: (3) function oscillates without staying near a single value

Example

Consider

$$f(x) = \sin(1/x)$$



$y = \sin(1/x)$

$\lim_{x\to 0} \sin(1/x)$ does not exist because there is no number L that the function $\sin(1/x)$ gets closer to as x approaches 0; indeed, the values of the function $\sin(1/x)$ oscillate more rapidly between -1 and 1 as x approaches 0.

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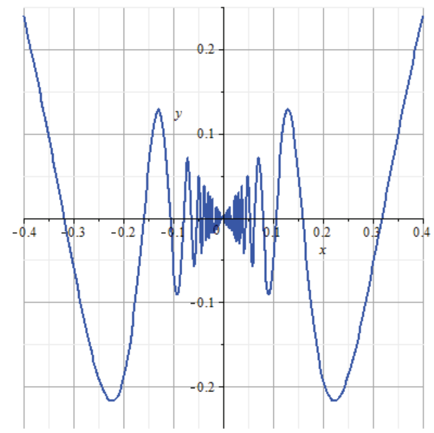
Example

On the other hand, consider

$$f(x) = x\sin(1/x)$$



$y = x\sin(1/x)$

$\lim_{x\to 0} x\sin(1/x)$ does exist because, even though the function oscillates infinitely often near $x = 0$, the function values are getting closer and closer to zero.

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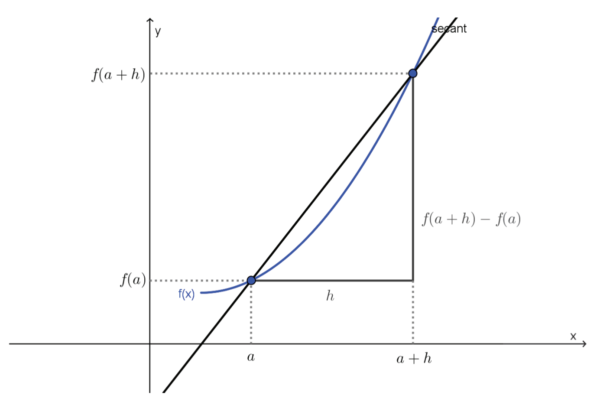
Limits of difference quotients

Remark

A *difference quotient* computes the slope of a secant line between two points $(a, f(a))$ and $(a+h, f(a+h))$:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

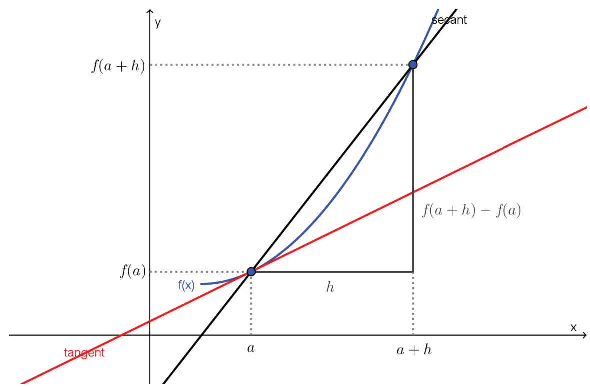


Limits of difference quotients

Remark

Taking the limit of these difference quotients as $h \to 0$ will give us the slope of the line that is tangent to $y = f(x)$ at the point $(a, f(a))$:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Link: GeoGebra tangent_line_slope.ggb

Just checking. . . .

1. True or false: The limit of $f(x)$ as x approaches 5 is $f(5)$.
2. Approximate $\lim_{x \to 0} f(x)$ numerically and graphically, where $f(x) = \begin{cases} \cos x & x \le 0 \\ x^2 + 3x + 1 & x > 0 \end{cases}$, or state why the limit does not exist.
3. Approximate $\lim_{x \to 2} \dfrac{x^2 + 7x + 10}{x^2 - 4x + 4}$ numerically and graphically, or state why the limit does not exist.
4. Approximate $\lim_{x \to 2} f(x)$ numerically and graphically, where $f(x) = \begin{cases} x + 1 & x < 2 \\ 3x - 5 & x \ge 2 \end{cases}$, or state why the limit does not exist.
5. Consider the function $f(x) = \ln x$ and the point $a = 5$. Approximate the limit of the difference quotient $\dfrac{f(a+h) - f(a)}{h}$ by using $h = \pm 0.1, \pm 0.01$.