1.2 Epsilon-delta definition of a limit

Toward a more rigorous definition

Definition

(Informal definitions) Given a function \( y = f(x) \), an \( x \)-value \( c \), and a
\( y \)-value \( L \), we say that \( \lim_{x \to c} f(x) = L \) provided

1. \( y = f(x) \) is near \( L \) whenever \( x \) is near \( c \).
2. whenever \( x \) is within a certain tolerance level of \( c \), then the
   corresponding value \( y = f(x) \) is within a certain tolerance level of
   \( L \).

Remark

The tolerances for \( x \) and \( y \) are different. The \( y \)-tolerance is called \( \varepsilon \),
and the \( x \)-tolerance is called \( \delta \).

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A rigorous definition of limit

Definition

(Rigorous definition) Let \( f \) be a function defined on an open interval
containing \( c \) (except perhaps at \( c \) itself). Then \( \lim_{x \to c} f(x) = L \) provided
that given any \( \varepsilon > 0 \), there exists a corresponding \( \delta > 0 \) such that
whenever \( 0 < |x - c| < \delta \), we have \( |f(x) - L| < \varepsilon \).
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Examples

Example

1. Show \( \lim_{x \to 9} \sqrt{x} = 3 \). (Cf. §1.2: Example 6)
2. Show \( \lim_{x \to 4} x^2 = 16 \). (Cf. §1.2: Example 7)
3. Show \( \lim_{x \to 0} e^x = 1 \). (Cf. §1.2: Example 9)
4. Show \( \lim_{x \to c} e^x = e^c \).

Remark
This last example shows that the function \( f(x) = e^x \) is continuous at all values of \( x \). More generally, a function \( f(x) \) is continuous at \( c \) if \( \lim_{x \to c} f(x) = f(c) \). We’ll explore this important idea of continuity more in §1.5.

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Just checking. . . .

1. What’s wrong with the following “definition” of a limit?
   The limit of \( f(x) \) as \( x \) approaches \( c \) is \( K \) means that given any \( \delta > 0 \), there exists an \( \varepsilon > 0 \) such that whenever \( |f(x) - K| < \varepsilon \) we have \( 0 < |x - c| < \delta \).
2. Using an \( \varepsilon - \delta \) argument, show \( \lim_{x \to 3} 5 = 5 \).
3. Using an \( \varepsilon - \delta \) argument, show \( \lim_{x \to 2} 3 - 2x = -1 \).
4. Using an \( \varepsilon - \delta \) argument, show \( \lim_{x \to 3} x^2 - 3 = 6 \).
5. Using an \( \varepsilon - \delta \) argument, show \( \lim_{x \to 0} e^{2x} - 1 = 0 \).