1.2 Epsilon-delta definition of a limit

Toward a more rigorous definition

Definition

(Informal definitions) Given a function $y = f(x)$, an x-value c, and a y-value L, we say that $\lim_{x \to c} f(x) = L$ provided

1. $y = f(x)$ is near L whenever x is near c.
2. whenever x is within a certain tolerance level of c, then the corresponding value $y = f(x)$ is within a certain tolerance level of L.

Remark

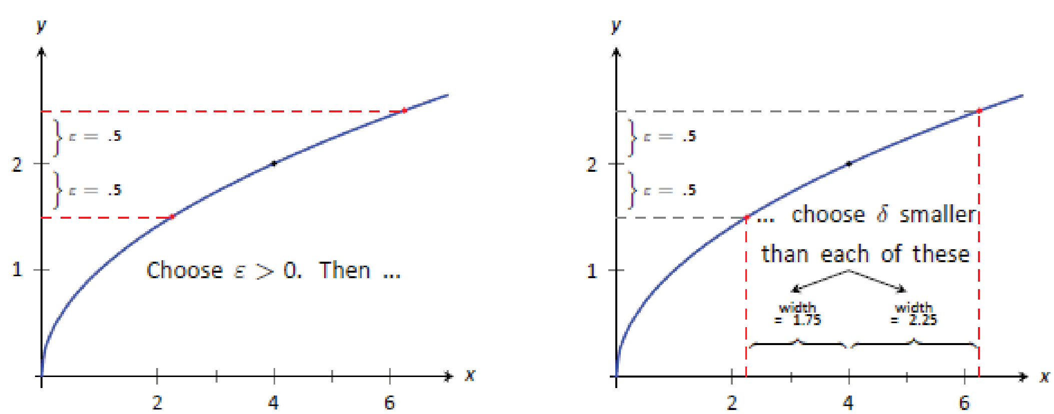
The tolerances for x and y are different. The y-tolerance is called ε, and the x-tolerance is called δ.

1.2 Epsilon-delta definition of a limit

A rigorous definition of limit

Definition

(Rigorous definition) Let f be a function defined on an open interval containing c (except perhaps at c itself). Then $\lim_{x \to c} f(x) = L$ provided that given any $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.



With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Examples

Example

1. Show $\lim_{x\to 9} \sqrt{x} = 3$. (Cf. §1.2: Example 6)
2. Show $\lim_{x\to 4} x^2 = 16$. (Cf. §1.2: Example 7)
3. Show $\lim_{x\to 0} e^x = 1$. (Cf. §1.2: Example 9)
4. Show $\lim_{x\to c} e^x = e^c$.

Remark

This last example show that the function $f(x) = e^x$ is *continuous* at all values of x. More generally, a function $f(x)$ is continuous at c if $\lim_{x\to c} f(x) = f(c)$. We'll explore this important idea of continuity more in §1.5.

Just checking. . . .

1. What's wrong with the following "definition" of a limit?

   > The limit of $f(x)$ as x approaches c is K means that given any $\delta > 0$, there exists an $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$ we have $0 < |x - c| < \delta$.

2. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 3} 5 = 5$.
3. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 2} 3 - 2x = -1$.
4. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 3} x^2 - 3 = 6$.
5. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 0} e^{2x} - 1 = 0$.