1.3 Finding limits analytically

Limit Laws

Theorem

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x \to c} f(x) = L \quad \lim_{x \to c} g(x) = K$$

Then the following limits hold:

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
4. $\lim_{x \to c} b \cdot f(x) = bL$
5. $\lim_{x \to c} f(x) \cdot g(x) = LK$
6. $\lim_{x \to c} f(x)/g(x) = L/K (K \neq 0)$
7. $\lim_{x \to c} [f(x)]^n = L^n$
8. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. And if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to L} g(x) = K$, then $\lim_{x \to c} g(f(x)) = K$.

1.3 Finding limits analytically

Using Limit Laws

Example

Suppose $\lim_{x \to 2} f(x) = 3$ and $\lim_{x \to 2} g(x) = -2$ and $p(x) = x^2 - 3x + 1$. Find:

1. $\lim_{x \to 2} f(x) + g(x) =$ __________
2. $\lim_{x \to 2} [g(x)]^3 =$ __________
3. $\lim_{x \to 2} p(x) =$ __________
4. $\lim_{x \to 2} f(x) - 3p(x) =$ __________

Other Limit Laws

Theorem

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x\to c} p(x) = p(c)$
2. $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, where $q(c) \neq 0$

Remark

Whenever $\lim_{x\to c} f(x) = f(c)$, we say that f is *continuous at* c. We will study continuity in greater detail in §1.5, but the following theorem says that each of the functions is continuous at every point in its domain.

Other Limit Laws

Theorem

Let c be a real number in the domain of the given function, let n be a positive integer, and let $a > 0$. Then the following limits hold:

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \csc x = \csc c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to c} a^x = a^c$
8. $\lim_{x\to c} \ln x = \ln c$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example

Evaluate the following limits:

1. $\lim_{x\to \pi/3} \cos x =$ ____________
2. $\lim_{x\to 3} \csc^2 x - \cot^2 x =$ ________
3. $\lim_{x\to 0} \ln e^{3x} =$ ____________
4. $\lim_{x\to 1} e^{\sqrt{4x}} =$ ____________

Squeeze theorem

Theorem

(Squeeze theorem) Let f, g and h be functions on an open interval I containing c such that for all x in I we have $f(x) \leq g(x) \leq h(x)$. If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$$

then

$$\lim_{x\to c} g(x) = L$$

Example

(A fundamental trig limit) $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

Remark

This says that $\sin x$ and x are approaching 0 at the same rate.

1.3 Finding limits analytically

Special limits

Theorem

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$
3. $\lim_{x\to 0} (1+x)^{1/x} = e$
4. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Remark

So, $\cos x - 1$ approaches 0 faster than x does, while $e^x - 1$ and x approach 0 at the same rate. These are all examples of *indeterminate forms*, which we will study in more detail in a later section (§6.7).

1.3 Finding limits analytically

Special limits

Example

Find the following limits:

1. $\lim_{x\to 0} \frac{\sin(3x)}{x} = \_\_\_\_\_\_\_\_\_\_$
2. $\lim_{x\to 0} \frac{\sin(5x)}{8x} = \_\_\_\_\_\_\_\_\_\_$
3. $\lim_{x\to 0} \frac{\tan(4x)}{3x} = \_\_\_\_\_\_\_\_\_\_$
4. $\lim_{x\to 0} x\csc(2x) = \_\_\_\_\_\_\_\_\_\_$

Remark

If numerator and denominator each contain a factor that is making the denominator zero, canceling the common factor can make it possible to evaluate the limit.

Example

Find $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.

1.3 Finding limits analytically

Using algebra to find limits

Theorem

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x\to c} g(x) = L$. Then

$$\lim_{x\to c} f(x) = L$$

Example

Find the following limits:

1. $\lim_{x\to 0} \frac{x^2+3x}{x^2-2x} = \_\_\_\_\_\_\_\_\_\_$
2. $\lim_{x\to -1} \frac{x^2+9x+8}{x^2-6x-7} = \_\_\_\_\_\_$
3. $\lim_{x\to 6} \frac{x^2-4x-12}{x^2-13x+42} = \_\_\_\_\_$
4. $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \_\_\_\_\_\_\_\_\_\_$
5. $\lim_{x\to 2} \frac{x^3-7x+6}{x^3-4x^2+x+6} = \_\_\_\_$
6. $\lim_{x\to 2} \frac{x^4-2x^3-7x+14}{x^4-16} = \_\_$

1.3 Finding limits analytically

Just checking. . . .

1. You are given the following information:
   - $\lim_{x\to 1} f(x) = 0 = \lim_{x\to 1} g(x)$
   - $\lim_{x\to 1} f(x)/g(x) = 2.$

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?
2. Find $\lim_{x\to 0} \dfrac{5x}{\cos(3x)}$.
3. Find $\lim_{x\to 0} \dfrac{5x}{\sin(3x)}$.
4. Find $\lim_{x\to \pi/4} \cos x \sin x$.
5. Find $\lim_{x\to -2} \dfrac{x^2 - 5x - 14}{x^2 + 10x + 16}$.
6. If $f(x) = x^2 - x$, find $\lim_{h\to 0} \dfrac{f(4+h) - f(4)}{h}$.

1.4 One-sided limits

Left- and right-hand limits

Definition

Let f be a function defined on an open interval containing c. The notation

$$\lim_{x\to c^-} f(x) = L,$$

which is read as "the limit of $f(x)$ as x approaches c from the left is L," or "the left-hand limit of f at c is L," means that:

> given any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < c - x < \delta$, we have $|f(x) - L| < \varepsilon$.

Remark

What changes need to be made to define a right-hand limit at c?