1.3 Finding limits analytically

Just checking. . . .

1. You are given the following information:
   - $\lim_{x\to 1} f(x) = 0 = \lim_{x\to 1} g(x)$
   - $\lim_{x\to 1} f(x)/g(x) = 2$.

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?
2. Find $\displaystyle\lim_{x\to 0} \frac{5x}{\cos(3x)}$.
3. Find $\displaystyle\lim_{x\to 0} \frac{5x}{\sin(3x)}$.
4. Find $\displaystyle\lim_{x\to \pi/4} \cos x \sin x$.
5. Find $\displaystyle\lim_{x\to -2} \frac{x^2 - 5x - 14}{x^2 + 10x + 16}$.
6. If $f(x) = x^2 - x$, find $\displaystyle\lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$.

1.4 One-sided limits

Left- and right-hand limits

Definition

Let f be a function defined on an open interval containing c. The notation

$$\lim_{x\to c^-} f(x) = L,$$

which is read as "the limit of $f(x)$ as x approaches c from the left is L," or "the left-hand limit of f at c is L," means that:

> given any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < c - x < \delta$, we have $|f(x) - L| < \varepsilon$.

Remark

What changes need to be made to define a right-hand limit at c?

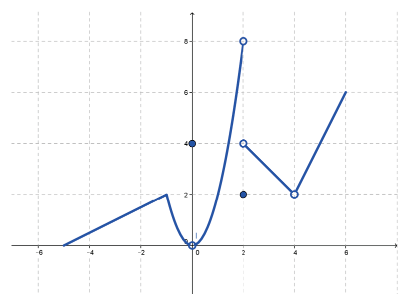
1.4 One-sided limits

Left- and right-hand limits

Example

The graph of the function $f(x)$ is shown below at right. Based on the graph, determine the following:

1. $\lim_{x\to -1^-} f(x) =$ ______________
2. $\lim_{x\to -1^+} f(x) =$ ______________
3. $\lim_{x\to -1} f(x) =$ ______________
4. $f(-1) =$ ______________
5. $\lim_{x\to 2^-} f(x) =$ ______________
6. $\lim_{x\to 2^+} f(x) =$ ______________
7. $\lim_{x\to 2} f(x) =$ ______________
8. $f(2) =$ ______________
9. $\lim_{x\to 0} f(x) =$ ______________
10. $f(0) =$ ______________
11. $\lim_{x\to 4} f(x) =$ ______________
12. $f(4) =$ ______________



1.4 One-sided limits

Left- and right-hand limits

Remark

From this example, we learn two important things about limits. First, $\lim_{x\to c} f(x)$ and $f(c)$ are independent of one another:

> $\lim_{x\to c} f(x)$ can be $f(c)$, but it need not be; in fact, $\lim_{x\to c} f(x)$ can exist even if $f(c)$ doesn't exist, and vice versa.

The second important thing we learn from this example is . . .

Theorem

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L$$

1.4 One-sided limits

Just checking. . . .

1. True or false: if $\lim_{x\to 5^-} f(x) = 3$, then $\lim_{x\to 5} f(x) = 3$.
2. True or false: if $\lim_{x\to 5} f(x) = 3$, then $\lim_{x\to 5^-} f(x) = 3$.
3. True or false: if $\lim_{x\to 5} f(x) = 3$, then $f(5) = 3$.
4. Estimate the limit numerically: $\lim_{x\to 0.2} \dfrac{x^2 + 5.8x - 1.2}{x^2 - 4.2x + 0.8}$
5. Evaluate the limit: $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x^3 - 3x}$

1.5 Continuity

Definition

Let f be a function defined on an open interval I containing c.

1. f is *continuous at* c if $\lim_{x\to c} f(x) = f(c)$.
2. f is continuous on I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is *continuous everywhere*.

Definition

Let f be defined on the closed interval $[a, b]$ for some real numbers a, b. f is *continuous on* $[a, b]$ if:

1. f is continuous on (a, b),
2. $\lim_{x\to a^+} f(x) = f(a)$, and
3. $\lim_{x\to b^-} f(x) = f(b)$.