1.4 One-sided limits

Just checking. . . .

1. True or false: if $\lim_{x\to 5^-} f(x) = 3$, then $\lim_{x\to 5} f(x) = 3$.
2. True or false: if $\lim_{x\to 5} f(x) = 3$, then $\lim_{x\to 5^-} f(x) = 3$.
3. True or false: if $\lim_{x\to 5} f(x) = 3$, then $f(5) = 3$.
4. Estimate the limit numerically: $\lim_{x\to 0.2} \dfrac{x^2 + 5.8x - 1.2}{x^2 - 4.2x + 0.8}$
5. Evaluate the limit: $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x^3 - 3x}$

1.5 Continuity

Definition

Let f be a function defined on an open interval I containing c.

1. f is *continuous at c* if $\lim_{x\to c} f(x) = f(c)$.
2. f is continuous on I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is *continuous everywhere*.

Definition

Let f be defined on the closed interval $[a, b]$ for some real numbers a, b. f is *continuous on* $[a, b]$ if:

1. f is continuous on (a, b),
2. $\lim_{x\to a^+} f(x) = f(a)$, and
3. $\lim_{x\to b^-} f(x) = f(b)$.

Example

Determine the domain of the given function and the interval(s) on which it is continuous.

Function	Domain	Interval(s) of continuity
$f(x) = 1/(x-2)$		
$f(x) = \sqrt{x-1}$		
$f(x) = \sqrt{4-x^2}$		
$f(x) = \lvert x\rvert/x$		

Properties of continuous functions

Theorem

Let f and g be continuous functions on an interval I, let c be a real number, and let n be a positive integer. The following functions are continuous on I.

1. $f \pm g$
2. $c \cdot f$
3. $f \cdot g$
4. f/g ($g \neq 0$ on I)
5. f^n
6. $\sqrt[n]{f}$
7. If the range of f is J and g is continuous on J, then $g \circ f$ is continuous on I.

1.5 Continuity

Common continuous functions

Theorem

The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \csc x$
5. $f(x) = \sec x$
6. $f(x) = \cot x$
7. $f(x) = \ln x$
8. $f(x) = a^x$ $(a > 0)$
9. $f(x) = \sqrt[n]{x}$ (n a positive integer)

Example

Find the interval(s) on which the following functions are continuous.

1. $f(x) = \sqrt{3 - x} + \sqrt{2x - 1}$ is continuous on ________________
2. $f(x) = \sqrt{\ln(2x + 3)}$ is continuous on ________________

1.5 Continuity

Intermediate value theorem

Theorem

(Intermediate value theorem) Let f be a continuous function on $[a, b]$. Then for every value y-value in between $f(a)$ and $f(b)$, there is an x-value c in $[a, b]$ such that $f(c) = y$.

Remark

The IVT is an existence theorem: it asserts the existence of an x-value c with $f(c) = y$ for every y in between $f(a)$ and $f(b)$. We can find a good approximation of c using the *bisection method*, as illustrated in the next example.

Example

(Bisection method) Find the root of $f(x) = x^3 + x + 1$ accurate to two decimal places.

Iteration	Interval	Midpoint sign
1	$[-1, 0]$	$f(-0.5) > 0$
2	$[-1, -0.5]$	$f(-0.75) < 0$
3	$[-0.75, -0.5]$	$f(-0.625) > 0$
4	$[-0.75, -0.625]$	$f(-0.6875) < 0$
5	$[-0.6875, -0.625]$	$f(-0.65625) > 0$
6	$[-0.6875, -0.65625]$	$f(-0.67188) > 0$
7	$[-0.6875, -0.67188]$	$f(-0.67969) > 0$
8	$[-0.6875, -0.67969]$	$f(-0.68359) < 0$
9	$[-0.68359, -0.67969]$	

Just checking. . . .

1. True or false: If f is defined on an open interval containing c and $\lim_{x\to c} f(x)$ exists, then f is continuous at c.
2. True or false: If f is continuous at c, then $\lim_{x\to c} f(x)$ exists.
3. True or false: If f is continuous at c, then $\lim_{x\to c} f(x) = f(c)$.
4. True or false: If f is continuous on $[0, 1)$ and on $[1, 2]$, then f is continuous on $[0, 2]$.
5. Let $f(x) = \left\{ \begin{array}{ll} x^2 & x \le 2 \\ 3 - mx & x > 2 \end{array} \right\}$. Find the value of m that makes f continuous at $x = 2$.