1.6 Limits involving infinity

Infinite limits

Definition

We say $\lim_{x \to c} f(x) = \infty$ if for every $M > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) \geq M$

Example

1. $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$
2. $\lim_{x \to \pi/2^-} \tan x = \infty$ and $\lim_{x \to \pi/2^+} \tan x = -\infty$.

Definition

If the limit of $f(x)$ as x approaches c from either the right or the left (or both) is ∞ or $-\infty$, we say that the function has a *vertical asymptote* at c.

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Example

1. $f(x) = \frac{x^2}{x^2 - 1}$ has vertical asymptotes at $x = 1$ and $x = -1$.
2. $f(x) = \frac{x^3 + 2x^2 - x - 2}{x^2 - 1}$ does not. Indeed, $f(x) = x + 2$ for all x except $x = 1$ and $x = -1$, where is simply has a hole (i.e. point discontinuity).

Remark

The above example illustrates the fact that just because a function has a denominator of zero for particular values of x does *not* mean that it has a vertical asymptote at those values of x. It must also have an infinite limit (from the left, or from the right, or both) in order to have a vertical asymptote.

Limits at infinity

Definition

1. We say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.
2. We say that $\lim_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.
3. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say that $y = L$ is a *horizontal asymptote* of f.

Example

Find the horizontal asymptote(s) of the following functions.

1. $f(x) = \dfrac{x^2}{x^2 - 1}$
2. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$
3. $f(x) = \dfrac{\sin x}{x}$

Limits at infinity

Theorem

Suppose we have a rational function

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

where any of the coefficients may be zero except for a_n and b_m.

1. If $n = m$, then $\lim_{x\to\pm\infty} f(x) = \dfrac{a_n}{b_m}$
2. If $n < m$, then $\lim_{x\to\pm\infty} f(x) = 0$.
3. If $n > m$, then $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are both infinite.

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Limits at infinity

Example

Find the following limits.

1. $\lim_{x\to\infty} \dfrac{6x-2x^2}{x^2-2x-3} =$ ________
2. $\lim_{x\to-\infty} \dfrac{6x-2x^3}{x^2-2x-3} =$ ________
3. $\lim_{x\to\infty} \dfrac{6x-2x^2}{x^3-2x-3} =$ ________
4. $\lim_{x\to 3} \dfrac{6x-2x^2}{x^2-2x-3} =$ ________

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Just checking. . . .

1. True or false: If $\lim_{x\to 3} f(x) = \infty$, then f has a vertical asymptote at $x = 3$.
2. True or false: If $\lim_{x\to 3} f(x) = \infty$, then $f(3)$ is not defined at $x = 3$.
3. True or false: If $\lim_{x\to 3} f(x) = \infty$, then f is not continuous at $x = 3$.
4. Using a $\varepsilon - \delta$ argument, show that $\lim_{x\to 1} 3x - 1 = 2$.
5. Evaluate the limit $\lim_{x\to -4} \dfrac{x^2-16}{x^2-4x-32}$.