2.1 The derivative

Rates of change

1. The slope of a secant line is

$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

and represents the *average rate of change* over $[a, b]$.
Letting $b = a + h$, we can express the slope of the secant line as

$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

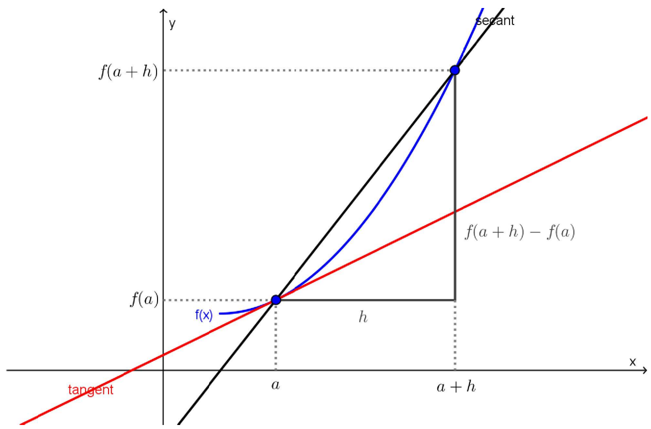
2. The slope of the tangent line to $y = f(x)$ at a is the limit of the secant slopes

$$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and represents the *instantaneous rate of change* at a.

2.1 The derivative

Rates of change



Link: GeoGebra tangent_line_slope.ggb

The derivative at a point

Definition

Let f be a continuous function on an open interval I and let c be in I. The *derivative* of f at c is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided the limit exists.

- If the limit exists we say that f is *differentiable at* c.
- If the limit does not exist, then f is *not differentiable at* c.
- If f is differentiable at every point in I, then f is *differentiable on* I.
- If f is differentiable at c, then $f'(c)$ is the slope of the line that is tangent to $y = f(x)$ at c, and $f'(c)$ represents the instantaneous rate of change in f at c.

Example

Let $f(x) = 2x^2 - 3x + 1$. Find

1. $f'(1) =$ ________________
   Tangent: ________________
2. $f'(3) =$ ________________
   Tangent: ________________
3. $f'(0) =$ ________________
   Tangent: ________________
4. $f'(-2) =$ ________________
   Tangent: ________________

and use your answer to write an equation for the tangent line to $y = f(x)$ at the given point.

Remark

An equation for a line of slope m through the point (x_0, y_0) may be written in either

- Point-slope form: $y - y_0 = m(x - x_0)$
- Slope-intercept form: $y = mx + b$

Normal lines

Remark

A line with slope m_1 is perpendicular to another line with slope m_2 if and only if $m_1 m_2 = -1$.

Definition

A *normal line* to $y = f(x)$ at c is a line that is perpendicular to the tangent line to $y = f(x)$ at c.

Remark

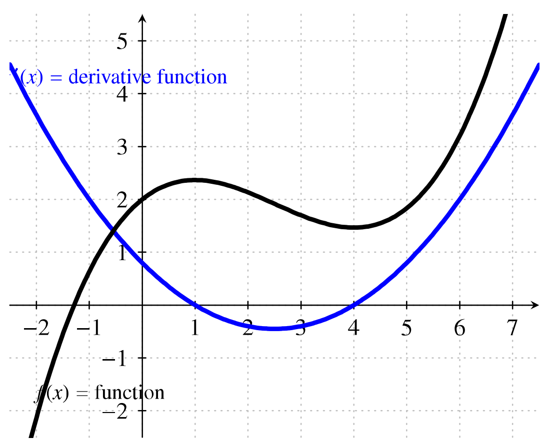
If $f'(c) \neq 0$, then the slope of the normal line is $-1/f'(c)$. If $f'(c) = 0$, then the normal line is the vertical line through $(c, f(c))$; that is, $x = c$.

Example

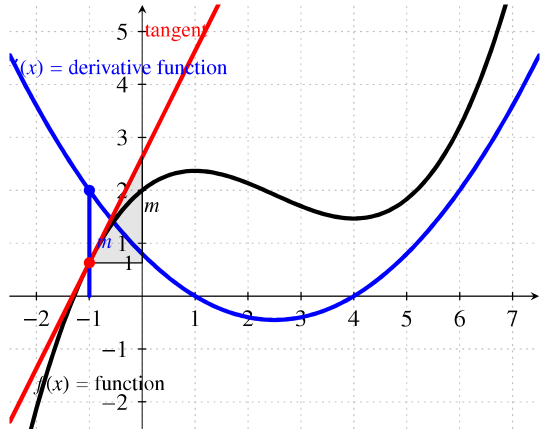
Find the normal lines to $f(x) = 2x^2 - 3x + 1$ at

- $x = 1$: ____________________
- $x = 0$: ____________________
- $x = 3$: ____________________
- $x = -2$: ____________________

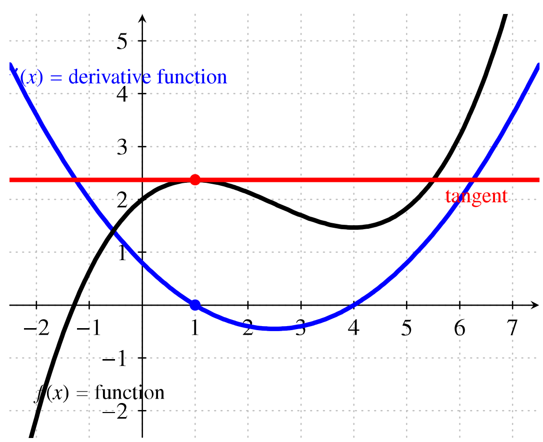
The derivative function



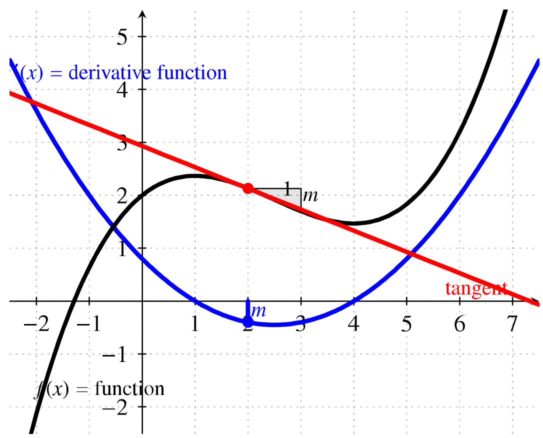
The derivative function



The derivative function



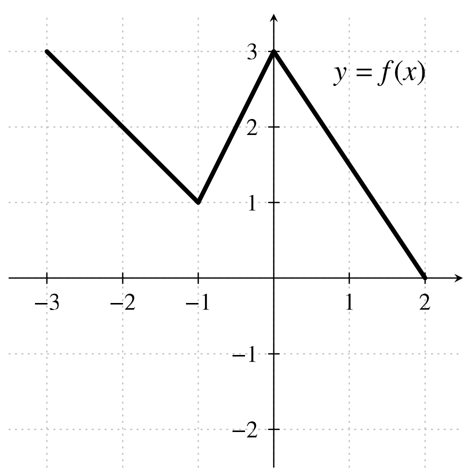
The derivative function



The derivative function

Example

Sketch the derivative of the following function.



The derivative function

Example

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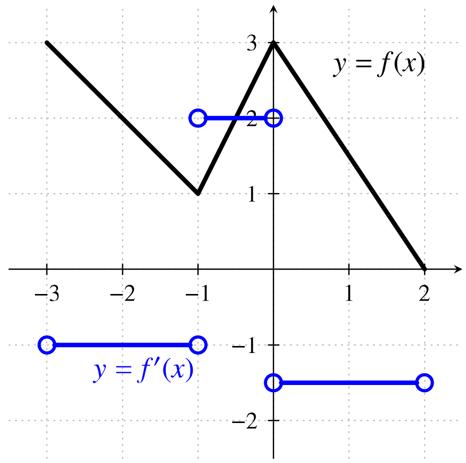


2.1 The derivative

The derivative function

Definition

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is the *derivative of* f.

If $y = f(x)$, then the following notations all represent the derivative:

$$\underbrace{f'(x) = y' = y'(x)}_{\text{Newton}} = \overbrace{\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f) = \frac{d}{dx}(y)}^{\text{Leibniz}}$$

Remark

In Leibniz notation $\frac{dy}{dx} = \lim_{h \to 0} \frac{\Delta y}{\Delta x}$.

The derivative function

Example

Find the derivative of each of the following functions.

1. $f(x) = 2x^2 - 3x + 1$
   $f'(x) =$ ________________
2. $g(x) = \dfrac{3}{x+2}$
   $g'(x) =$ ________________
3. $h(x) = 3x - 2$
   $h'(x) =$ ________________
4. $k(x) = \sqrt{3x}$
   $k'(x) =$ ________________

Remark

1. Evaluating the derivative function $f'(x)$ at $x = c$ gives us $f'(c)$, which is the slope of the tangent line at c.
2. The line that is tangent to a linear function is the line itself. If $a(x) = |x|$, what is $a'(x)$?

The derivative function

Example

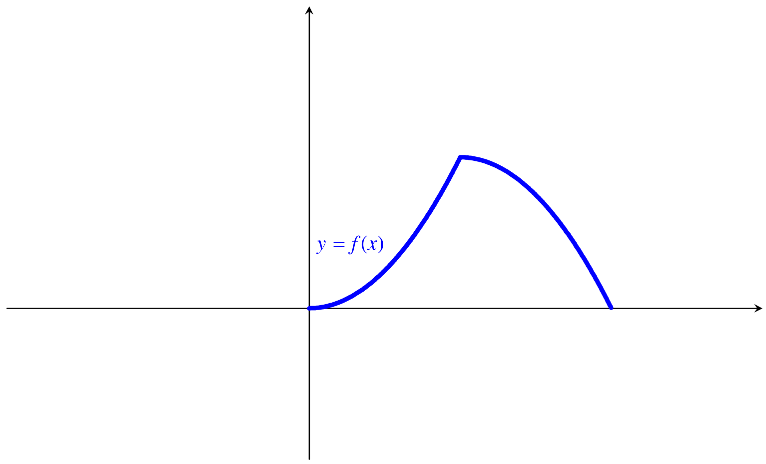
Find the derivative of each of the following functions.

1. $f(x) = 2x^2 - 3x + 1$
   $f'(x) = 4x - 3$
2. $g(x) = \dfrac{3}{x+2}$
   $g'(x) = \dfrac{-3}{(x+2)^2}$
3. $h(x) = 3x - 2$
   $h'(x) = 3$
4. $k(x) = \sqrt{3x}$
   $k'(x) = \dfrac{3}{2\sqrt{3x}}$

Remark

1. Evaluating the derivative function $f'(x)$ at $x = c$ gives us $f'(c)$, which is the slope of the tangent line at c.
2. The line that is tangent to a linear function is the line itself. If $a(x) = |x|$, what is $a'(x)$?

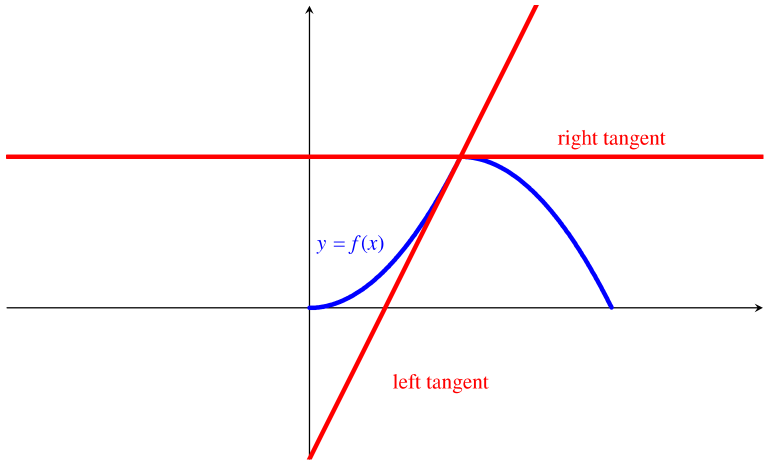
Differentiability



A function is not differentiable at a corner, where the one-sided tangent lines have different slopes.

2.1 The derivative

Differentiability



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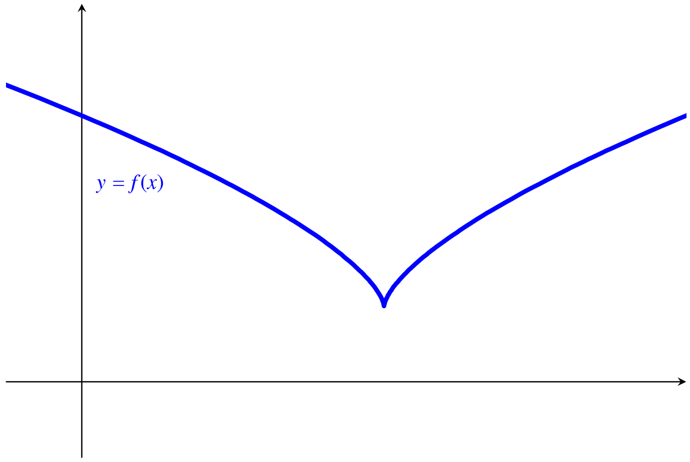
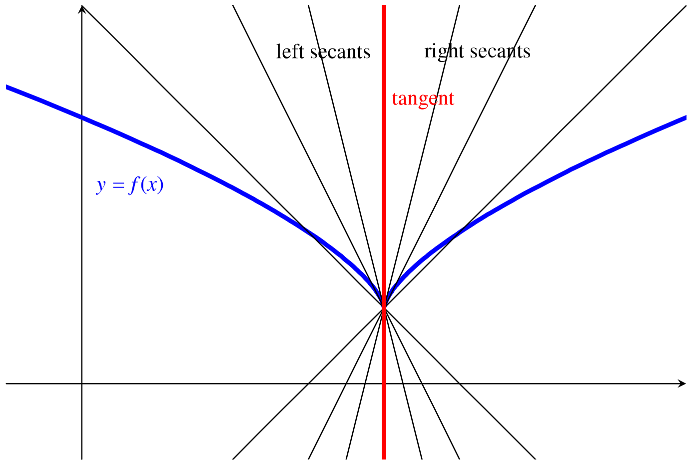
Differentiability



A function is not differentiable at a cusp, where the secant slopes approach $-\infty$ from one side and $+\infty$ from the other.

Differentiability



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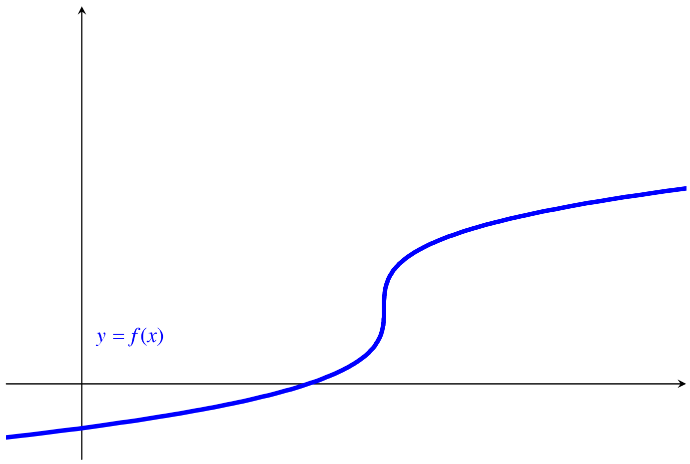
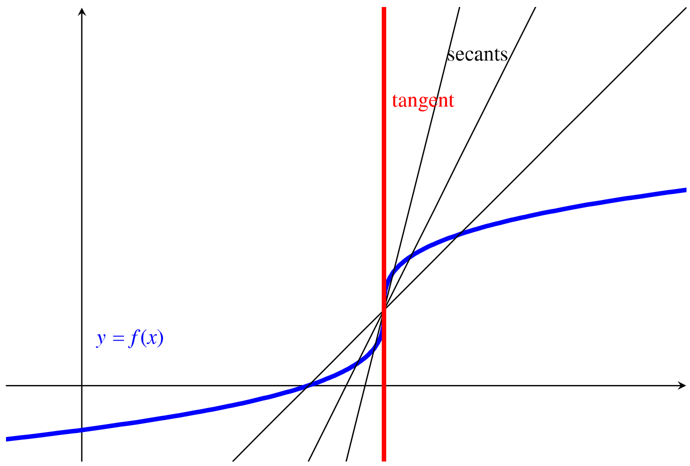
Differentiability



A function is not differentiable at a vertical tangent, where the secant slopes approach $+\infty$ from both sides (or $-\infty$ from both sides).

Differentiability



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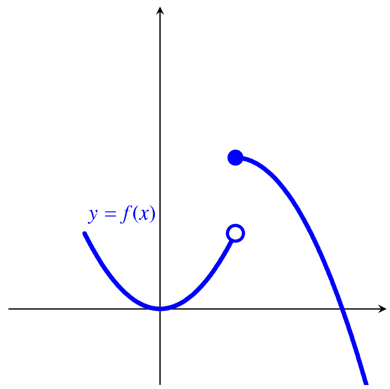
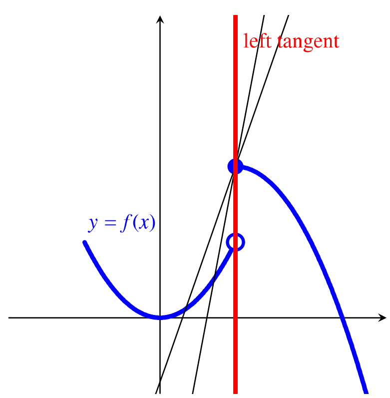
Differentiability



A function is not differentiable at a jump discontinuity, where the one-sided tangents have different (and possibly infinite) slopes

2.1 The derivative

Differentiability



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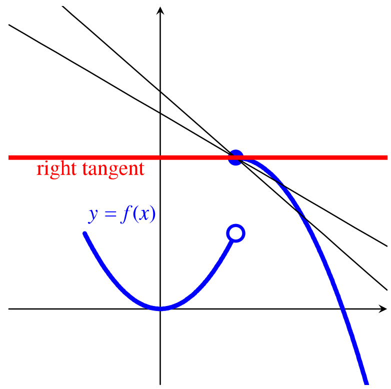
Differentiability



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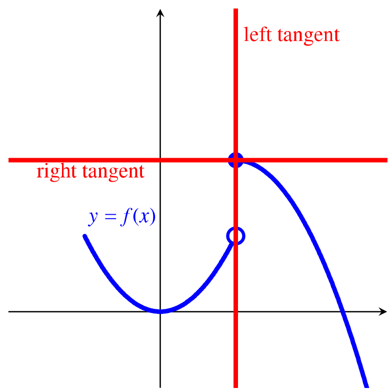
2.1 The derivative

Differentiability



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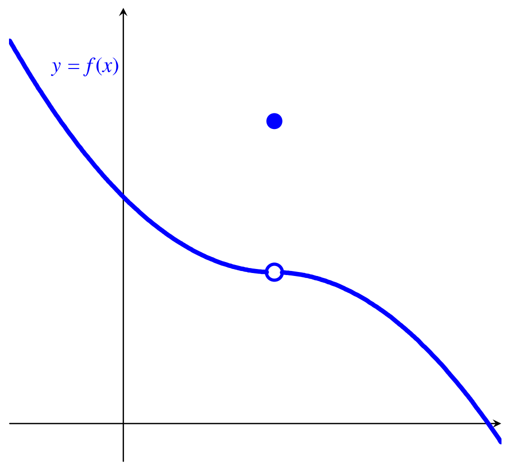
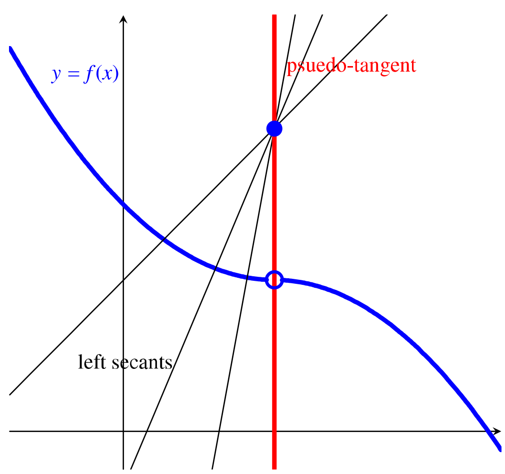
Differentiability



A function is not differentiable at a removable discontinuity, where the secant slopes approach $+\infty$ from one side and $-\infty$ from the other

2.1 The derivative

Differentiability



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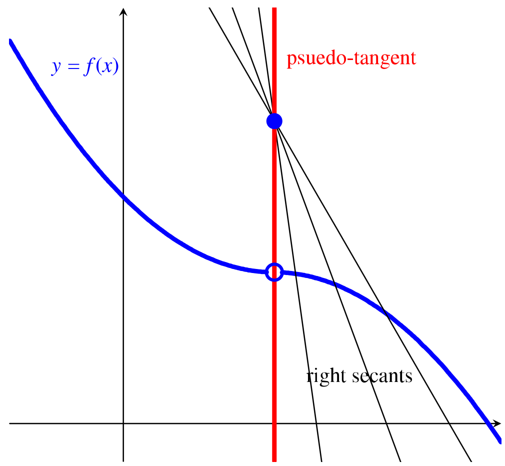
Differentiability



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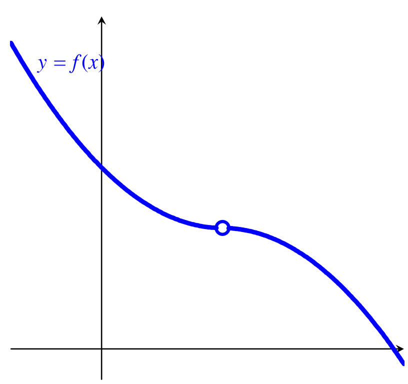
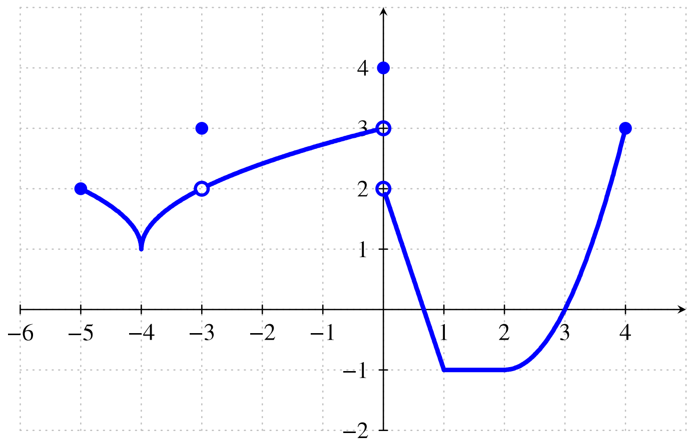
Differentiability



A function is not differentiable at a removable discontinuity where $f(a)$ does not exist since $f(a)$ is required for the tangent slope computation $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

Differentiability

- Find the x in $[-5, 4]$ for which $y = f(x)$ is **not** continuous.
- Find the x in $[-5, 4]$ for which $y = f(x)$ is **not** differentiable.
- Estimate $f'(3)$.



2.1 The derivative

Differentiability implies continuity

Theorem

If $f(x)$ has a derivative at $x = a$, then $f(x)$ is continuous at $x = a$.

Note: the converse is **not** true. There are continuous functions that are not differentiable.

Differentiability implies continuity

Claim: If $f'(a)$ exists, then f is continuous at a.

Proof: For $x \neq a$,

$$f(x) = f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limit as $x \to a$,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a} \right).$$

2.1 The derivative

Differentiability implies continuity

Claim: If $f'(a)$ exists, then f is continuous at a.

Proof: For $x \neq a$,

$$f(x) = f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limit as $x \to a$,

$$\lim_{x \to a} f(x) = f(a) + \lim_{x \to a} (x - a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2.1 The derivative

Differentiability implies continuity

Claim: If $f'(a)$ exists, then f is continuous at a.

Proof: For $x \neq a$,

$$f(x) = f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limit as $x \to a$,

$$\lim_{x \to a} f(x) = f(a) + 0 \cdot f'(a).$$

2.1 The derivative

Differentiability implies continuity

Claim: If $f'(a)$ exists, then f is continuous at a.

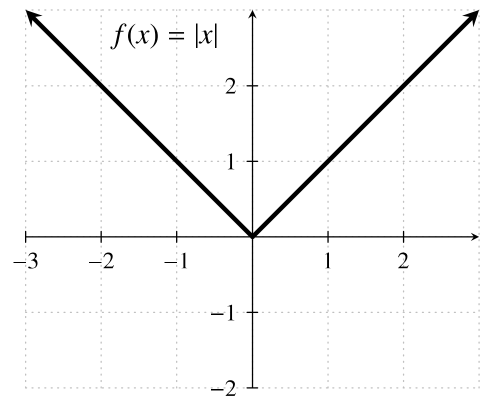
Proof: For $x \neq a$,

$$f(x) = f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limit as $x \to a$,

$$\lim_{x \to a} f(x) = f(a).$$

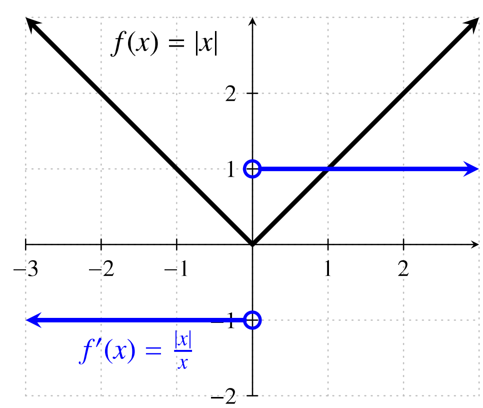
Continuous but not differentiable



Corner at $x = 0$.

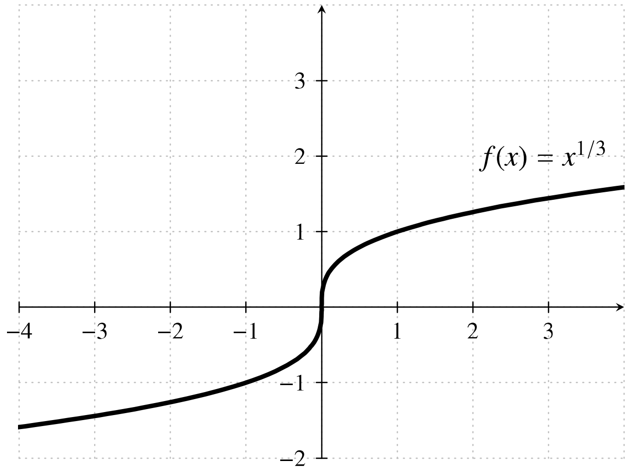
2.1 The derivative

Continuous but not differentiable



Corner at $x = 0$.

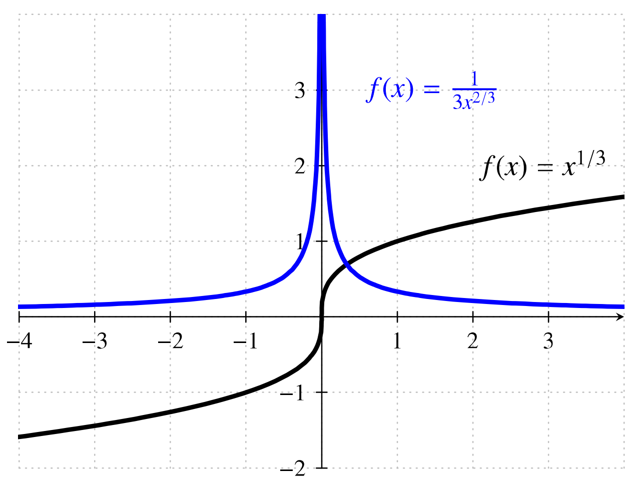
Continuous but not differentiable



Vertical tangent at $x = 0$.

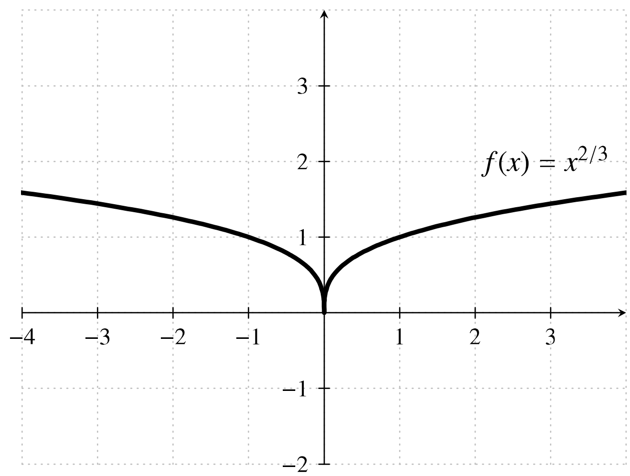
2.1 The derivative

Continuous but not differentiable



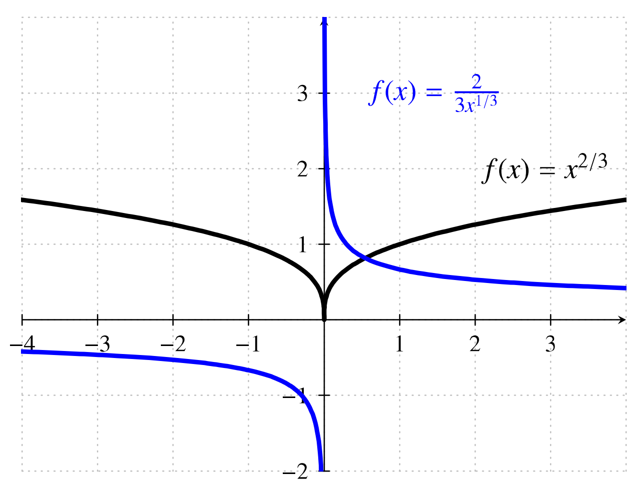
Vertical tangent at $x = 0$.

Continuous but not differentiable



Cusp and vertical tangent at $x = 0$.

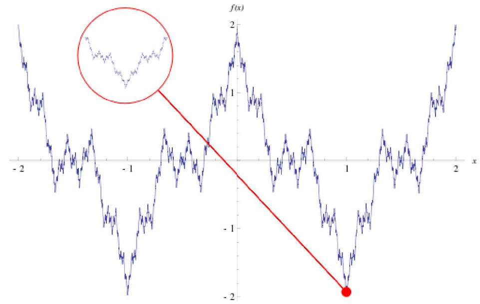
Continuous but not differentiable



Cusp and vertical tangent at $x = 0$.

2.1 The derivative

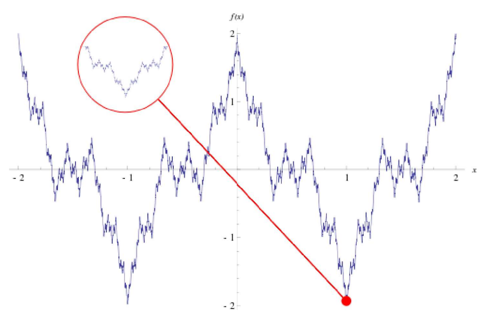
Weierstrass Function Karl Weierstrass constructed a function that is continuous everywhere but differentiable nowhere.



Plot of a Weierstrass Function over the interval $[-2, 2]$. Like fractals, the function exhibits self-similarity: every zoom (red circle) is similar to the global plot.

2.1 The derivative

Weierstrass Function Karl Weierstrass constructed a function that is continuous everywhere but differentiable nowhere.



$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$.

Basic trig derivatives

Remark

Recall the angle addition formulas from trigonometry:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

and the limits we encountered in chapter 1:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

These are the key ingredients in the proof of the following theorem.

Theorem

- $\boxed{\dfrac{d}{dx}(\sin x) = \cos x}$
- $\boxed{\dfrac{d}{dx}(\cos x) = -\sin x}$

Example

Find the derivative of $f(x) = \begin{cases} \cos x & x \geq 0 \\ 1 & x < 0 \end{cases}$

Remark

1. A continuous function can be described as one whose graph we could sketch without lifting our pencil.
2. A differentiable function can be described as a continuous function that does not have any "sharp corners."

Just checking. . . .

1. Find the derivative of $f(x) = \dfrac{1}{x}$.
2. Find equations of the tangent and normal lines to $y = 1/x$ at $(2, 1/2)$.
3. Approximate the value of the derivative $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ for $f(x) = e^x$ by taking $h = 0.1$.
4. The approximation above is an [overestimate | underestimate] of the true value of $f'(0)$ for $f(x) = e^x$. (Hint: think graphically)
5. Sketch the graph of the derivative of the function shown at right.