2.3 Differentation rules!

Example

Using a graph of the function (whenever possible) or the defintion of the derivative (whenever necesary), find the derivatives of the following functions.

1. $f(x) = x^0$
   $f'(x) =$ ______________
2. $f(x) = x$
   $f'(x) =$ ______________
3. $f(x) = x^2$
   $f'(x) =$ ______________
4. $f(x) = x^3$
   $f'(x) =$ ______________
5. $f(x) = 1/x$
   $f'(x) =$ ______________
6. $f(x) = 3x$
   $f'(x) =$ ______________
7. $f(x) = 3x + 1$
   $f'(x) =$ ______________
8. $f(x) = e$
   $f'(x) =$ ______________

2.3 Differentation rules!

Example

Using a graph of the function (whenever possible) or the defintion of the derivative (whenever necesary), find the derivatives of the following functions.

1. $f(x) = x^0$
   $f'(x) = 0$
2. $f(x) = x$
   $f'(x) = 1$
3. $f(x) = x^2$
   $f'(x) = 2x$
4. $f(x) = x^3$
   $f'(x) = 3x^2$
5. $f(x) = 1/x$
   $f'(x) = -1/x^2$
6. $f(x) = 3x$
   $f'(x) = 3$
7. $f(x) = 3x + 1$
   $f'(x) = 3$
8. $f(x) = e$
   $f'(x) = 0$

Theorem

Basic differentiation rules

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(e^x) = e^x$
6. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Theorem

Basic differentiation properties

1. $\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$
2. $\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot f'(x)$

Remark

Differentiation respects addition and constant multiples because derivatives are limits (of difference quotients) and limits respect addition and constant multiples:

- $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} c \cdot f(x) = c \cdot \lim_{x\to a} f(x)$

But derivatives do not inherit all the properties of limits (as we'll see in the next section) because those limit laws concerned limits of functions, whereas derivatives are limits of difference quotients. For instance, the limit of a product (or quotient) is the product (or quotient) of the limits, but the derivative of a product (or quotient) is *not* the product (or quotient) of the derivatives (as we'll see in the next section).

Example

Find the derivatives of the following functions.

1. $f(x) = 2x^2 - 3x + 1$
   $f'(x) =$ ________________
2. $g(x) = 3e^x + 2\sin x$
   $g'(x) =$ ________________

Find an equation for the line tangent to

1. f at $x = 3$
   ____
2. g at $x = 0$
   ____

Now

1. Without using any calculus approximate $g(0.1) \approx$ ________
2. Approximate $g(0.1)$ by using an appropriate tangent line.
   $g(0.1) \approx$ ________________________

2.3 Differentation rules!

Example

Find the derivatives of the following functions.

1. $f(x) = 2x^2 - 3x + 1$
   $f'(x) = 4x - 3$
2. $g(x) = 3e^x + 2\sin x$
   $g'(x) = 3e^x + 2\cos x$

Find an equation for the line tangent to

1. f at $x = 3$
   $y - 10 = 9(x - 3)$
2. g at $x = 0$
   $y - 3 = 5x$

Now

1. Without using any calculus approximate $g(0.1) \approx g(0) = 3$
2. Approximate $g(0.1)$ by using an appropriate tangent line.
   $g(0.1) \approx 3.5$ (Cf. $g(0.1) = 3.515$)

Example

Let $f(x) = \cos x + x/2 - 1$.

1. Approximate $f(3)$ without using any calculus.

   $f(3) \approx$ ____________________

2. Now approximate $f(3)$ by using an appropriate tangent line.

   $f(3) \approx$ ____________________

3. Find the value(s) of x, if any, where f has a horizontal tangent.

   $x =$ ____________________

Example

Let $f(x) = \cos x + x/2 - 1$.

1. Approximate $f(3)$ without using any calculus.

   $f(3) \approx f(\pi) = \pi/2 - 2 \approx -0.43$

2. Now approximate $f(3)$ by using an appropriate tangent line.

   $f(3) \approx -0.5$ (Cf. $f(3) = -0.490$)

3. Find the value(s) of x, if any, where f has a horizontal tangent.

   $x = \ldots, -7\pi/6, \pi/6, 5\pi/6, \ldots$ (i.e. whenever $\sin x = 1/2$)

Higher order derivatives

Definition

Let $y = f(x)$ be differentiable on an interval I.

1. The *second derivative* of f is

$$f''(x) = \frac{d}{dx}\big(f'(x)\big) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''$$

2. The *third derivative* of f is

$$f'''(x) = \frac{d}{dx}\big(f''(x)\big) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''$$

3. The n^{th} *derivative* of f is

$$f^{(n)}(x) = \frac{d}{dx}\big(f^{(n-1)}(x)\big) = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx}\right) = \frac{d^ny}{dx^n} = y^{(n)}$$

2.3 Differentation rules!

Higher order derivatives

Example

Find the first four derivatives of the following functions.

1. $f(x) = 3x^2 - 2x$

   $f'(x) =$ ________

   $f''(x) =$ ________

   $f'''(x) =$ ________

   $f^{(4)}(x) =$ ________

2. $f(x) = 3e^x$

   $f'(x) =$ ________

   $f''(x) =$ ________

   $f'''(x) =$ ________

   $f^{(4)}(x) =$ ________

3. $f(x) = \cos x$

   $f'(x) =$ ________

   $f''(x) =$ ________

   $f'''(x) =$ ________

   $f^{(4)}(x) =$ ________

Higher order derivatives

Example

Find the first four derivatives of the following functions.

1. $f(x) = 3x^2 - 2x$
   $f'(x) = 6x - 2$
   $f''(x) = 6$
   $f'''(x) = 0$
   $f^{(4)}(x) = 0$

2. $f(x) = 3e^x$
   $f'(x) = 3e^x$
   $f''(x) = 3e^x$
   $f'''(x) = 3e^x$
   $f^{(4)}(x) = 3e^x$

3. $f(x) = \cos x$
   $f'(x) = -\sin x$
   $f''(x) = -\cos x$
   $f'''(x) = \sin x$
   $f^{(4)}(x) = \cos x$

Higher order derivatives

Remark

The second derivative is the rate of change of the rate of change of the function; or, put geometrically, the rate of change of the slope of tangent lines.

- The sign of the first derivative tells us whether the function is changing positively (i.e. increasing) or negatively (i.e. decreasing). So, the sign of the first derivative tells us *whether* the function is increasing or decreasing.
- The sign of the second derivative tells us whether the tangent slopes are changing positively (i.e increasing) or negatively (i.e. decreasing). So, the sign of the first derivative tells us *how* the function is increasing or decreasing.

2.3 Differentation rules!

Just checking. . . .

1. Differentiate whichever functions you can using the differentiation rules we've discussed so far.

   a. $f(x) = 3/x^2$
   b. $g(x) = 3/(x+1)^2$
   c. $h(x) = (3x^2+1)/x^3$
   d. $i(x) = 3x^3/(x^3+2)$
   e. $j(x) = \sqrt[3]{x}$
   f. $k(x) = \sqrt{x+1}$
   g. $\ell(x) = \sqrt[4]{x} + 3$
   h. $m(x) = 2e^x$
   i. $n(x) = e^{2x}$
   j. $p(x) = xe^2$
   k. $q(x) = \ln(x^2)$
   l. $r(x) = 2\sin x$
   m. $s(x) = \sin(2x)$
   n. $t(x) = \sin x \cos x$

2. Where does the line that is tangent to $f(x) = e^x + 3$ at $x = 0$ intersect the x-axis?
3. Approximate $e^{0.1}$ using an appropriate tangent line.

2.4 The product and quotient rules

Theorem

(The Product Rule)

Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Example

Find the derivatives of the following functions.

1. $f(x) = 3x\sin x$

   $f'(x) =$ ____________________

2. $f(x) = xe^x$

   $f'(x) =$ ____________________

3. $f(x) = x\ln x - x$

   $f'(x) =$ ____________________

4. $f(x) = (x+1)(3x^2-2)$

   $f'(x) =$ ____________________