2.4 The product and quotient rules

Just checking. . . .

1. True or false. The derivatives of the trigonometric "co-" functions (i.e. the ones that start with "co-") have minus signs in them.
2. Find the values of x in $[-1, 1]$ where the tangent line to $f(x) = x \sin x$ is horizontal.
3. Find an equation of the normal line to $f(x) = \frac{x^2}{x-1}$ at $(2, 4)$.
4. Find the derivative of $xe^x \sin x$.
5. Find the derivative of $\sin x \csc x$.

2.5 The chain rule

Example

Find the derivatives of the following functions.

1. $F_2(x) = (1 - x)^2$. $F_2'(x) =$ ______________________________
2. $F_3(x) = (1 - x)^3$. $F_3'(x) =$ ______________________________
3. $F_4(x) = (1 - x)^4$. $F_4'(x) =$ ______________________________

Remark

Notice that each of the functions is a composition $F_n(x) = f_n(g(x))$, where the "inner piece" is $g(x) = 1 - x$ and the "outer piece" is $f_n(x) = x^n$ (for $n = 2, 3, 4$). Notice, too, that we can differentiate each "piece." The chain rule tells us how to put the derivatives of these pieces together to get the derivative of a composition.

Example

Find the derivatives of the following functions.

1. $F_2(x) = (1-x)^2$. $F_2'(x) = -2(1-x)$
2. $F_3(x) = (1-x)^3$. $F_3'(x) = -3(1-x)^2$
3. $F_4(x) = (1-x)^4$. $F_4'(x) = -4(1-x)^3$

Remark

Notice that each of the functions is a composition $F_n(x) = f_n(g(x))$, where the "inner piece" is $g(x) = 1 - x$ and the "outer piece" is $f_n(x) = x^n$ (for $n = 2, 3, 4$). Notice, too, that we can differentiate each "piece." The chain rule tells us how to put the derivatives of these pieces together to get the derivative of a composition.

Theorem

(The Chain Rule)

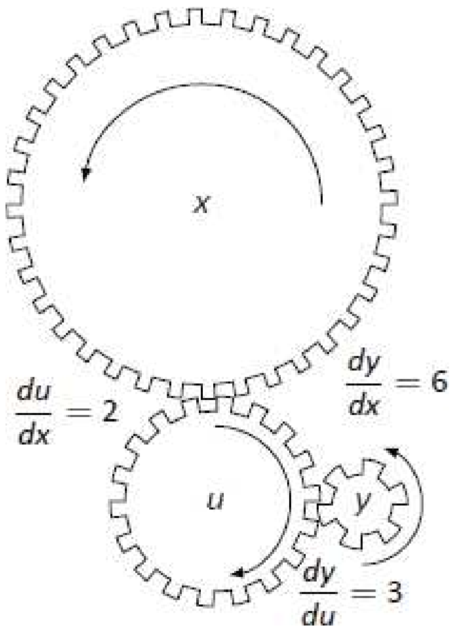
Let $y = f(u)$ be a differentiable function of u, and let $u = g(x)$ be a differentiable function of x.

Then $y = f(g(x))$ is a differentiable function of x and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

or in Newton's notation

$$\boxed{\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)}$$



Example

Find the derivatives of the following functions.

1. $y = \cos(3x)$

   $dy/dx =$ ____________

2. $y = \sin^{17} x$

   $dy/dx =$ ____________

3. $y = \ln(x^{-2})$

   $dy/dx =$ ____________

4. $y = e^{x^2}$

   $dy/dx =$ ____________

Theorem

Let $u = u(x)$ be a differentiable function of x. Then:

1. $\frac{d}{dx}(u^n) = nu^{n-1} \cdot (du/dx)$
2. $\frac{d}{dx}(e^u) = e^u \cdot (du/dx)$
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot (du/dx)$
4. $\frac{d}{dx}(\sin u) = \cos u \cdot (du/dx)$
5. $\frac{d}{dx}(\cos u) = -\sin u \cdot (du/dx)$
6. $\frac{d}{dx}(\tan u) = \sec^2 u \cdot (du/dx)$

2.5 The chain rule

Example

Find the derivatives of the following functions.

1. $y = \cos(3x)$

   $dy/dx = -3\sin(3x)$

2. $y = \sin^{17} x$

   $dy/dx = 17\sin^{16} x \cos x$

3. $y = \ln(x^{-2})$

   $dy/dx = -2x$

4. $y = e^{x^2}$

   $dy/dx = 2xe^{x^2}$

Theorem

Let $u = u(x)$ be a differentiable function of x. Then:

1. $\frac{d}{dx}(u^n) = nu^{n-1} \cdot (du/dx)$
2. $\frac{d}{dx}(e^u) = e^u \cdot (du/dx)$
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot (du/dx)$
4. $\frac{d}{dx}(\sin u) = \cos u \cdot (du/dx)$
5. $\frac{d}{dx}(\cos u) = -\sin u \cdot (du/dx)$
6. $\frac{d}{dx}(\tan u) = \sec^2 u \cdot (du/dx)$

2.5 The chain rule

Example

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin(3x)$

   $f'(x) =$ ____________

2. $f(x) = \dfrac{3x+1}{\sin^3 x}$

   $f'(x) =$ ____________

3. $f(x) = \cos(\sqrt{1-2x})$

   $f'(x) =$ ____________

4. $f(x) = \ln(2e^{\sin x})$

   $f'(x) =$ ____________

Remark

Recall that $e^{\ln u} = u$. So $a^x = e^{\ln a^x} = e^{x \ln a}$.

2.5 The chain rule

Example

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin(3x)$

   $f'(x) =$

   $5x^4 \sin(3x) + 3x^5 \cos(3x)$

2. $f(x) = \dfrac{3x+1}{\sin^3 x}$

   $f'(x) =$

   $\dfrac{3 \sin^3 x - 3x \sin^3 x - \sin^3 x}{\sin^6 x}$

3. $f(x) = \cos(\sqrt{1-2x})$

   $f'(x) = \dfrac{\sin \sqrt{1-2x}}{\sqrt{1-2x}}$

4. $f(x) = \ln(2e^{\sin x})$

   $f'(x) = \cos x$

Remark

Recall that $e^{\ln u} = u$. So $a^x = e^{\ln a^x} = e^{x \ln a}$.

Theorem

Let $a > 0$ (and $a \neq 1$). Then

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

More generally, if $u = u(x)$ is a differentiable function of x, then

$$\frac{d}{dx}(a^u) = a^u(\ln u) \cdot \frac{du}{dx}$$

Also,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

and more generally

$$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \cdot \frac{du}{dx}$$

2.5 The chain rule

Just checking. . . .

1. Compute $\frac{d}{dx}(\ln(kx))$ in two ways: (a) by using the chain rule, and (b) by using laws of logs first, and then differentiating.
2. Compute $\frac{d}{dx}\left(\ln(x^k)\right)$ in two ways: (a) by using the chain rule, and (b) by using laws of logs first, and then differentiating.
3. True or false. $\frac{d}{dx}(3^x) \approx (1.1)3^x$.
4. Compute $\frac{d}{dx}\left(\cos(1/x)e^{5x^2}\right)$.
5. Find equations for the tangent and normal lines to $g(x) = (\sin x + \cos x)^3$ at $x = \pi/2$.