2.7 Derivatives of inverse functions

Background

Definition

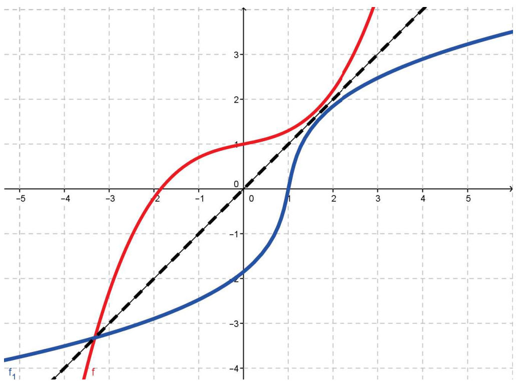
Two functions f and g are inverses of each other provided

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

We write $f^{-1}(x)$ for the inverse of f.

Remark

If g is the inverse of f, then g "undoes" whatever f does: if $f(a) = b$, then $g(b) = a$. A consequence of this is that the graph of the inverse function $y = f^{-1}(x)$ is the reflection of the graph of the function $y = f(x)$ through the line $y = x$.



2.7 Derivatives of inverse functions

Background

Remark

A function takes each input to a single output: $f(a) = b$. Since the inverse function will take b back to a, we need f to take *only one* input to b; otherwise, an inverse function cannot be defined.

Definition

A function f is *injective* (or *one-to-one*) if distinct inputs get sent to distinct outputs:

- If $a \neq b$, then $f(a) \neq f(b)$

  – or equivalently –

- If $f(a) = f(b)$, then $a = b$.

Background

Theorem

A function f has an inverse if and only if f is injective.

Remark

If f is injective, then a horizontal line will intersect the graph of f at most once, and vice versa.

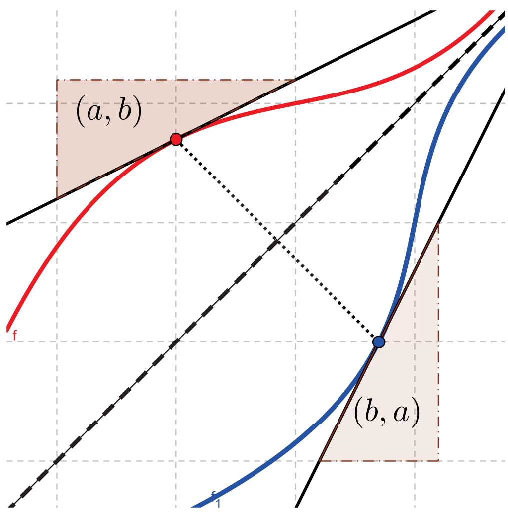
Example

Given that $y = \dfrac{2x-3}{x+1}$ is injective, find the inverse function. (Can you show that this function is injective?)

2.7 Derivatives of inverse functions

Remark

Recall that a function and its inverse are reflections of each other through the line $y = x$. A consequence of this is that the derivative of the inverse at a point is the reciprocal of the derivative of the function at the corresponding point.



2.7 Derivatives of inverse functions

Theorem

Let f be a differentiable and injective function, and let $g = f^{-1}$ be the inverse of f. Suppose $f(a) = b$ so that $g(b) = a$. Then

$$\left(f^{-1}\right)'(b) = g'(b) = \frac{1}{f'(a)}$$

and more generally

$$\left(f^{-1}\right)'(x) = g'(x) = \frac{1}{f'(g(x))}$$

Example

$g(x) = \arcsin x$ and $f(x) = \sin x$ are inverses, and so the theorem above gives

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)} \qquad (???)$$

2.7 Derivatives of inverse functions

Inverse trig functions

Inverse trig functions (sans domain restriction) are defined by:

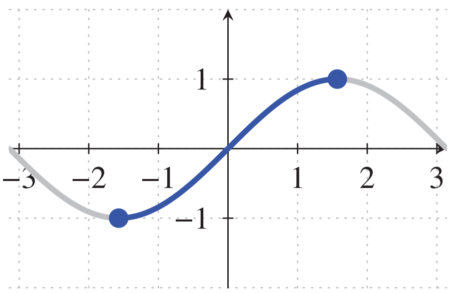
$$\begin{aligned} y = \arcsin(x) &\quad\Leftrightarrow\quad \sin(y) = x. \\ y = \arccos(x) &\quad\Leftrightarrow\quad \cos(y) = x. \\ y = \arctan(x) &\quad\Leftrightarrow\quad \tan(y) = x. \end{aligned}$$

Caution: Inverse trig functions are not reciprocal functions.

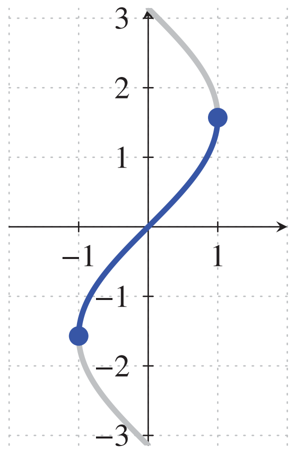
$$\begin{aligned} \arcsin(x) = \operatorname{asin}(x) = \sin^{-1}(x) &\quad\neq\quad \tfrac{1}{\sin(x)} = \csc(x). \\ \arccos(x) = \operatorname{acos}(x) = \cos^{-1}(x) &\quad\neq\quad \tfrac{1}{\cos(x)} = \sec(x). \\ \arctan(x) = \operatorname{atan}(x) = \tan^{-1}(x) &\quad\neq\quad \tfrac{1}{\tan(x)} = \cot(x). \end{aligned}$$

2.7 Derivatives of inverse functions

Arcsine



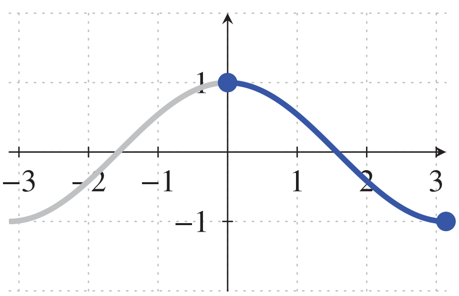
$y = \sin(x)$
$-\pi/2 \leq x \leq \pi/2$
$-1 \leq y \leq 1$



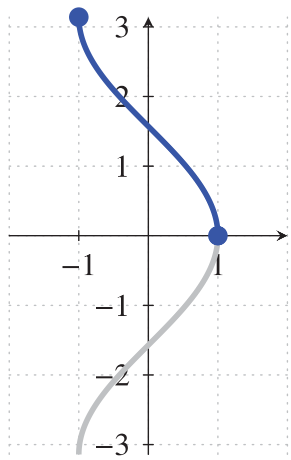
$y = \arcsin(x)$
$-1 \leq x \leq 1$
$-\pi/2 \leq y \leq \pi/2$

2.7 Derivatives of inverse functions

Arccosine

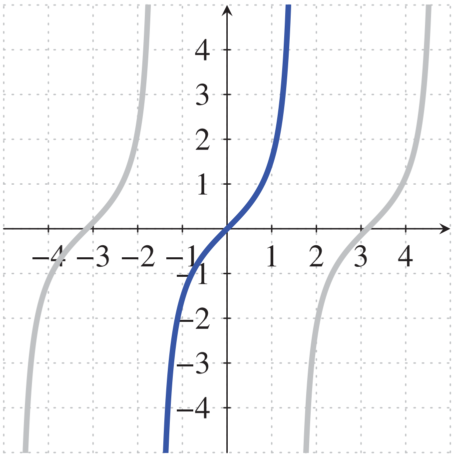


$y = \cos(x)$
$0 \leq x \leq \pi$
$-1 \leq y \leq 1$

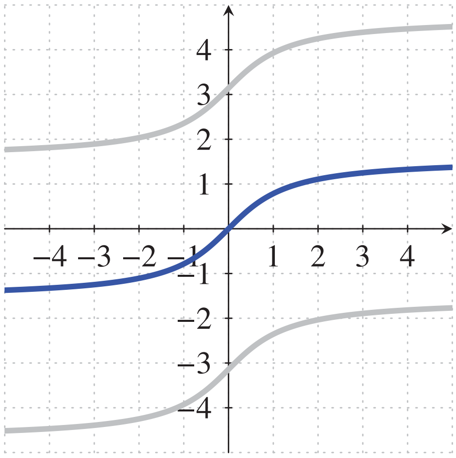


$y = \arccos(x)$
$-1 \leq x \leq 1$
$0 \leq y \leq \pi$

Arctangent



$y = \tan(x)$
$-\pi/2 < x < \pi/2$
$-\infty < y < \infty$



$y = \arctan(x)$
$-\infty < x < \infty$
$-\pi/2 < y < \pi/2$

Derivatives of inverse trig functions

Example

Find $\frac{d}{dx}(\arccos(3x))$.

Example

Find $\frac{d}{dx}\left(\arcsin(2x^3)\right) =$ ______________________________

Remark

This technique can be used to show $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

2.7 Derivatives of inverse functions

Derivatives of inverse trig functions

Example

Find $\frac{d}{dx}(\arccos(3x))$.

Example

Find $\frac{d}{dx}\left(\arcsin(2x^3)\right) = \dfrac{6x^2}{\sqrt{1-4x^6}}$

Remark

This technique can be used to show $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

2.7 Derivatives of inverse functions

Just checking. . . .

1. If $(3,4)$ lies on the graph of $y = f(x)$ and $f'(3) = -5$, what can be said about the graph of $y = f^{-1}(x)$?
2. Find an equation of the line that is tangent to $x^2 + y^2 + xy = 7$ at the point $(1,2)$.
3. Compute the derivative of the function $f(x) = \sin(\cos^{-1} x)$ in two ways:
   a. by using the chain rule first, and then simplifying
   b. by simplifying first, and then taking the derivative
4. Find an equation of the line that is tangent to the function $f(x) = \sin^{-1}(2x)$ at $x = 1/4$.
5. Find the derivative of $y = \dfrac{(x+3)^7(x-2)^3}{\sqrt[3]{2x-5}}$.