Definition

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum** (or **absolute minimum**) of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum** (or **absolute maximum**) of f on I if $f(c) \geq f(x)$ for all x in I.

The maximum and minimum values are the **extreme values** (or **extrema**) of f on I. We also call absolute extrema **global extrema**.

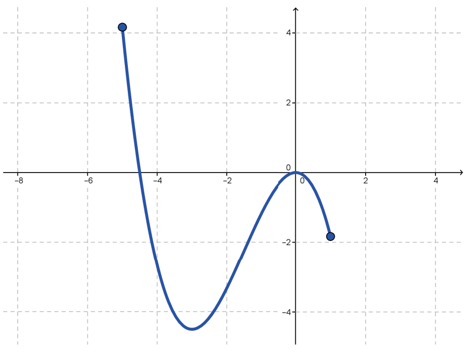
Theorem

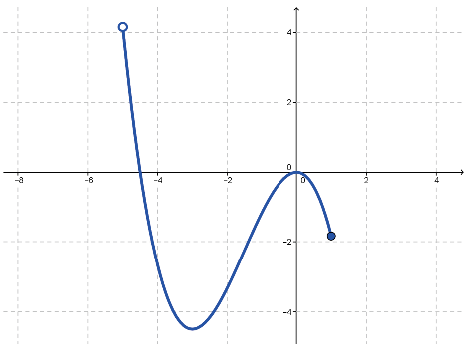
Extreme Value Theorem

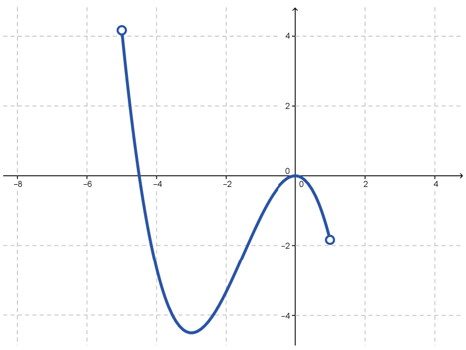
Let f be a continuous function defined on a closed interval I. Then f has both a maximum and a minimum value on I.

3.1 Extreme values

Example







Definition

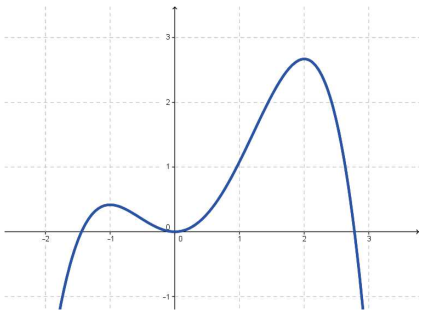
Let f be defined on an interval I containing c.

1. If there is an open interval containing c such that $f(c)$ is the minimum value on that interval, then $f(c)$ is a **relative minimum** or **local minimum** of f.
2. If there is an open interval containing c such that $f(c)$ is the maximum value on that interval, then $f(c)$ is a **relative maximum** or **local maximum** of f.

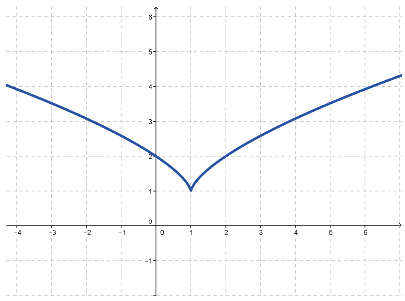
Collectively, relative maxima and minima are called **relative extrema** or **local extrema**.

Example

1. $f(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2$



2. $f(x) = (x-1)^{2/3} + 1$



Definition

Let f be defined at c. The x-value c is a **critical number** or **critical value** of f if either

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

If c is a critical value of f, the point $(c, f(c))$ is a **critical point** of f.

Theorem

If f has a relative extreme value at c, then c is a critical number of f.

Remark

That is, all relative extrema occur at critical values. However, not all critical values give relative extrema.

Remark

Finding extrema on closed intervals

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$ (whose existence is guaranteed by the Extreme Value Theorem), we:

1. Find the critical numbers of f in $[a, b]$.
2. Evaluate f at the critical numbers and at the endpoints.
3. Compare values to find the greatest and least.

Example

Find the extreme values of the following functions on the given intervals, or, if no interval is given, on its domain.

1. $f(x) = x^2 + x + 4$ on $[-1, 2]$
   max: ____________________
   min: ____________________
2. $f(x) = (\ln x)/x$ on $[1, 4]$
   max: ____________________
   min: ____________________
3. $f(x) = e^x \sin x$ on $[0, \pi]$
   max: ____________________
   min: ____________________
4. $f(x) = x^2 \sqrt{4 - x^2}$
   max: ____________________
   min: ____________________

3.1 Extreme values

Just checking. . . .

1. True or false. If c is a critical value of a function f, then f has either a relative maximum or a relative minimum at $x = c$.
2. Find the extreme values of $f(x) = x^3 - \frac{9}{2}x^2 - 30x + 3$ on $[0, 6]$.
3. Let $f(x) = x^3 + x$. Evaluate $\lim\limits_{s \to 0} \dfrac{f(x+s) - f(x)}{s}$.
4. Find $\frac{d}{dx}\left(\sin^{-1}(e^{2x})\right)$.
5. Find $\frac{d}{dx}\left(3^{x^2}\right)$.

3.2 The mean value theorem

Remark

- The average rate of change of f over the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

- The instantaneous rate of change in f at c is $f'(c)$.

Theorem

The Mean Value Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a value of c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$