3.3 Increasing and decreasing functions

Definition

Let f be a function on an interval I.

1. f is **increasing** on I if $f(a) \leq f(b)$ for every $a < b$ in I.
2. f is **decreasing** on I if $f(a) \geq f(b)$ for every $a < b$ in I.

We say that f is *strictly increasing* (or *strictly decreasing*) on I if the inequalities are strict.

Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for every c in (a, b), then f is increasing on $[a, b]$.
2. If $f'(c) < 0$ for every c in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(c) = 0$ for every c in (a, b), then f is constant on $[a, b]$.

3.3 Increasing and decreasing functions

Remark

Finding intervals on which f is increasing or decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing or decreasing:

1. Find the critical values of f; that is, the c in I where either $f'(c) = 0$ or $f'(c)$ is undefined.
2. Use the critical values to divide I into subintervals.
3. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   a. If $f'(p) > 0$, then f is increasing on that subinterval.
   b. If $f'(p) < 0$, then f is decreasing on that subinterval.

3.3 Increasing and decreasing functions

Example

Find the intervals on which $f(x) = \frac{x^2+15}{x-1}$ is increasing and the intervals on which it is decreasing.

- f is increasing on ________________________________
- f is decreasing on ________________________________

Theorem

First Derivative Test

Let f be differentiable on I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If the sign of f' does not change of c, then $f(c)$ is not a relative minimum or relative maximum of f.

3.3 Increasing and decreasing functions

Example

Find the intervals on which $f(x) = \frac{x^2+15}{x-1}$ is increasing and the intervals on which it is decreasing.

- f is increasing on $(-\infty, -3) \cup (5, \infty)$
- f is decreasing on $(-3, 5)$

Theorem

First Derivative Test

Let f be differentiable on I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If the sign of f' does not change of c, then $f(c)$ is not a relative minimum or relative maximum of f.

3.3 Increasing and decreasing functions

Example

1. Find and classify the critical points of $f(x) = \frac{x^2+15}{x-1}$.
   - Local minima ____________________
   - Local maxima ____________________
   - Neither ____________________
2. Find and classify the critical points of $f(x) = \frac{(x-2)^{2/3}}{x}$.
   - Local minima ____________________
   - Local maxima ____________________
   - Neither ____________________
3. Find and classify the critical points of $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.
   - Local minima ____________________
   - Local maxima ____________________
   - Neither ____________________

3.3 Increasing and decreasing functions

Example

1. Find and classify the critical points of $f(x) = \frac{x^2+15}{x-1}$.
   - Local minima $f(5) = 10$
   - Local maxima $f(-3) = -6$
   - Neither $x = 1$
2. Find and classify the critical points of $f(x) = \frac{(x-2)^{2/3}}{x}$.
   - Local minima $f(2) = 0$
   - Local maxima $f(6) = 4^{2/3}/6$
   - Neither $x = 0$
3. Find and classify the critical points of $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.
   - Local minima $f(-3\pi/4) = f(\pi/4) = 1/2$
   - Local maxima $f(-\pi/4) = f(3\pi/4) = -1/2$
   - Neither *none*

3.3 Increasing and decreasing functions

Just checking. . . .

1. Find and classify the critical points of $f(x) = 2x^3 + x^2 - x + 3$.
2. Find the value of c in $(-1, 2)$ where the instantaneous rate of change in $f(x) = x^2 - 3x + 5$ at c is equal to the average rate of chenge in f over $[-1, 2]$.
3. Evaluate $\lim_{x\to 5} \dfrac{x/5 - 5/x}{x - 5}$.
4. Using an $\varepsilon - \delta$ argument, show that $\lim_{x\to 2} 3x^2 = 12$.
5. Find the points on $x^2 - xy + y^2 = 1$ where the tangent line is horizontal and where it is vertical.

3.4 Concavity and the second derivative

Definition

The graph of a function f on an interval

- is **concave up** if it lies above its tangent lines
- is **concave down** if it lies below its tangent lines
- has no concavity if it is flat (i.e., linear)