3.3 Increasing and decreasing functions

Just checking. . . .

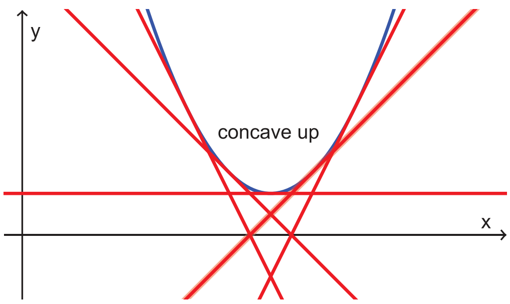
1. Find and classify the critical points of $f(x) = 2x^3 + x^2 - x + 3$.
2. Find the value of c in $(-1, 2)$ where the instantaneous rate of change in $f(x) = x^2 - 3x + 5$ at c is equal to the average rate of chenge in f over $[-1, 2]$.
3. Evaluate $\lim_{x \to 5} \dfrac{x/5 - 5/x}{x - 5}$.
4. Using an $\varepsilon - \delta$ argument, show that $\lim_{x \to 2} 3x^2 = 12$.
5. Find the points on $x^2 - xy + y^2 = 1$ where the tangent line is horizontal and where it is vertical.

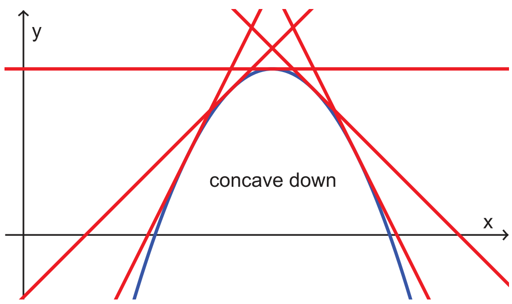
3.4 Concavity and the second derivative

Definition

The graph of a function f on an interval

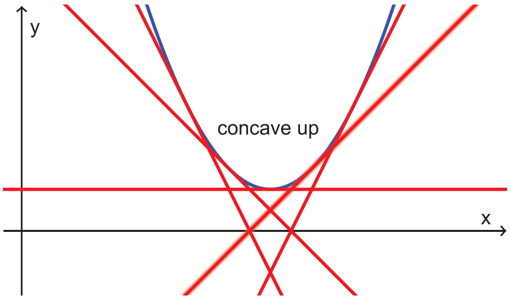
- is **concave up** if it lies above its tangent lines
- is **concave down** if it lies below its tangent lines
- has no concavity if it is flat (i.e., linear)





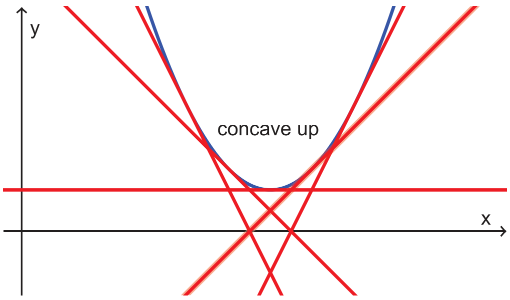
3.4 Concavity and the second derivative

What is happening with the slopes of the tangents when f is concave up? What does this mean for the derivatives of f?



3.4 Concavity and the second derivative

What is happening with the slopes of the tangents when f is concave up? What does this mean for the derivatives of f?



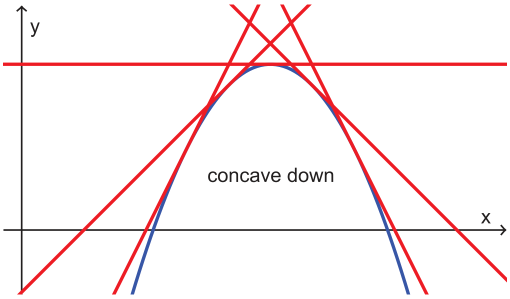
When f is concave up on an interval:

- the slopes of the tangents are increasing.
- $f'(x)$ is increasing.
- $f''(x) > 0$.

Reality check: try $f(x) = x^2$.

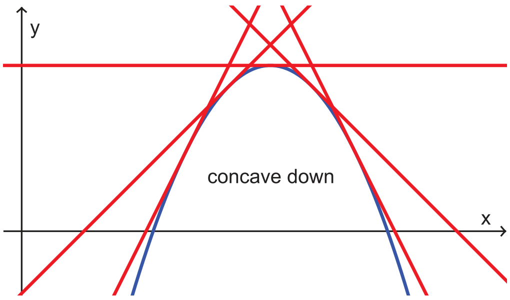
3.4 Concavity and the second derivative

What is happening with the slopes of the tangents when f is concave down? What does this mean for the derivatives of f?



3.4 Concavity and the second derivative

What is happening with the slopes of the tangents when f is concave down? What does this mean for the derivatives of f?



When f is concave down on an interval:

- the slopes of the tangents are decreasing.
- $f'(x)$ is decreasing.
- $f''(x) < 0$.

Reality check: try $f(x) = -x^2$.

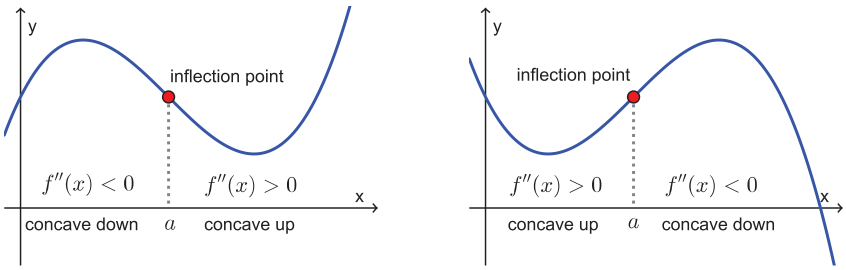
3.4 Concavity and the second derivative

Theorem

Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I and concave down if $f'' < 0$ on I.

Definition

A **point of inflection** is a point on the graph of f at which the concavity of f changes.



Theorem

If $(c, f(c))$ is a point of inflection, then either $f''(c) = 0$ or $f''(c)$ is not defined.

3.4 Concavity and the second derivative

Example

Find the x-coordinates of inflection points (IPs) of the following functions.

1. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x$

   IPs ____________________

2. $f(x) = \frac{x}{x^2-1}$

   IPs ____________________

3. $f(x) = x^2 e^x$

   IPs ____________________

4. $f(x) = x^2 \ln x$

   IPs ____________________

Theorem

Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

Example

Find the x-coordinates of inflection points (IPs) of the following functions.

1. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x$
   IPs $x = 1/3, 1$
2. $f(x) = \frac{x}{x^2-1}$
   IPs $x = -1, 0, 1$
3. $f(x) = x^2 e^x$
   IPs $x = -2 \pm \sqrt{2}$
4. $f(x) = x^2 \ln x$
   IPs $x = e^{-3/2}$

Theorem

Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

Just checking. . . .

1. Is it possible for a function to be
   - increasing and concave down on $(0, \infty)$ with a horizontal asymptote of $y = 1$?
   - increasing and concave up on $(0, \infty)$ with a horizontal asymptote of $y = 1$?
2. Find and classify the critical points of $f(x) = x^2 e^x$.
3. Find the inflection points of $f(x) = x^2 e^x$.
4. Find and classify the critical points of $f(x) = x^2 \ln x$.
5. Find the inflection points of $f(x) = x^2 \ln x$.