4.3 Optimization

Method

1. Draw a picture and label relevant variables and constants.
2. Identify the quantity to be minimized or maximized and the constraint.
3. Use the constraint to write the quantity to be optimized as a function of a single variable, and find the extreme values using the first derivative.

Example

1. Find the maximum product of two numbers that have a sum of 100.
2. Find the maximum sum of two numbers in $[0, 60]$ whose product is 100.
3. Find the maximum area of a right triangle whose hypotenuse has length 1.

Example

1. You are walking along the beach at Tunnel Park with your dog, who can run about 22 ft/s and swim about 1 ft/s. You throw a stick into the lake 20 feet down the shore line and 15 feet into the water.
   How far along the shore should your dog run before jumping into the lake in order to minimize the time it takes to get the stick?
2. The United States Postal Service charges more for boxes whose combined length and girth exceeds 108 inches. (The length of a package is the length of its longest side; the girth is the perimeter of the cross section.)
   What is the maximum volume of a package with a square cross section that does not exceed the 108-inch standard?

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Example

1. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.
2. A right triangle whose hypotenuse is $\sqrt{3}$ meters long is revolved about one of its legs to generate a right circular cone. Findthe radius, height , and volume of the cone of greatest volume that can be made this way.
3. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10. How about the same problem with an inscribed cone?

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Just checking. . . .

1. Use an $\varepsilon - \delta$ argument to show that $\lim_{x\to 2} x^2 - 3 = 1$.
2. Use the definition of the derivative to show that $\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$.

   (Hint: $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.)
3. Find $\lim_{x\to 0} \dfrac{\tan(2x)}{3x}$.
4. Find the maximum and minimum values of $f(x) = x^3 + 3x^2 - 2$ on $[-3, 2]$.
5. Find the value(s) of c where the instantaneous rate of change at c equals the average rate of change over $[-3, 2]$ for the function $f(x) = x^3 + 3x^2 - 2$.