5.1 Antiderivatives and indefinite integration

Definition

Let $f(x)$ be a function. An **antiderivative** of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x)$$

The set of all antiderivatives of $f(x)$ is the **indefinite integral** of f, denoted by

$$\int f(x)\, dx$$

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Theorem

If $f'(x) = 0$ for all x on an interval I, then f is constant on I.

Proof.

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Theorem

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C$$

Proof.

Remark

Thus, any two antiderivatives of a function differ only by an additive constant, and so we commonly write

$$\int f(x)\,dx = F(x) + C$$

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Example

1. $\int \sin x\,dx =$ ____________________
2. $\int e^x\,dx =$ ____________________
3. $\int \frac{1}{x}\,dx =$ ____________________
4. $\int 3x^2 + 4x - 1\,dx =$ ____________________
5. $\int 0\,dx =$ ____________________
6. $\int \quad dx =$ ____________________

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Example

1. $\int \sin x \, dx =$
   $-\cos x + C$
2. $\int e^x \, dx =$
   $e^x + C$
3. $\int \frac{1}{x} \, dx =$
   $\ln x + C$[1]
4. $\int 3x^2 + 4x - 1 \, dx =$
   $x^2 + 2x^2 - x + C$
5. $\int 0 \, dx =$
   C
6. $\int \, dx =$
   $x + C$

Remark

By checking our answers, we see that differentiating "undoes" integrating:

$$\frac{d}{dx}\left(\int f(x) \, dx\right) = f(x)$$

[1] To make the domains agree, we should write $\int (1/x) \, dx = \ln|x| + C$.

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Theorem 35 **Derivatives and Antiderivatives**

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}(cf(x)) = c \cdot f'(x)$	1. $\int c \cdot f(x) \, dx = c \cdot \int f(x) \, dx$
2. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$	2. $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
3. $\frac{d}{dx}(C) = 0$	3. $\int 0 \, dx = C$
4. $\frac{d}{dx}(x) = 1$	4. $\int 1 \, dx = \int dx = x + C$
5. $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$	5. $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}(\sin x) = \cos x$	6. $\int \cos x \, dx = \sin x + C$
7. $\frac{d}{dx}(\cos x) = -\sin x$	7. $\int \sin x \, dx = -\cos x + C$
8. $\frac{d}{dx}(\tan x) = \sec^2 x$	8. $\int \sec^2 x \, dx = \tan x + C$
9. $\frac{d}{dx}(\csc x) = -\csc x \cot x$	9. $\int \csc x \cot x \, dx = -\csc x + C$
10. $\frac{d}{dx}(\sec x) = \sec x \tan x$	10. $\int \sec x \tan x \, dx = \sec x + C$
11. $\frac{d}{dx}(\cot x) = -\csc^2 x$	11. $\int \csc^2 x \, dx = -\cot x + C$
12. $\frac{d}{dx}(e^x) = e^x$	12. $\int e^x \, dx = e^x + C$
13. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$	13. $\int a^x \, dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}(\ln x) = \frac{1}{x}$	14. $\int \frac{1}{x} \, dx = \ln|x| + C$

Remark

1. From this table we can see that differentiation and integration are inverse operations: each one "undoes" the other. Inverse operations do the opposite things in opposite order.
2. Antidifferentiation produces a family of functions: if F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C$$

So antiderivatives are determined up to an additive constant. We can determine the additive constant when we have some additional information about the antiderivative.

Initial value problems

Example

1. Find a solution to $\dfrac{dy}{dx} = x^3 + \dfrac{3}{x}$ if $y = 6.25$ when $x = 1$.
2. If $f''(x) = 24x^2 + 12x - 8$ and $f(0) = 5$ and $f(1) = 1$, find $f(x)$.
3. A car braked with a constant deceleration of 40 feet per second squared, producing skid marks measuring 160 feet before coming to a stop. How fast was the car traveling when the brakes were first applied?
4. A car is traveling 80 feet per second (approximately 55 miles per hour) when the brakes are fully applied, producing a constant deceleration of 40 feet per second squared. What is the distance traveled between when the brakes are first applied and when the car comes to a stop?

5.1 Antiderivatives and indefinite integration

Just checking. . . .

1. Find $\int (x^2 + 3)(x - 2)\, dx$.
2. Given $y = x^2 e^x \cos x$, find dy.
3. Find dy/dx if $x^3 + xy - \cos y = 5$.
4. Find the area of the largest rectangle that can be placed between the x-axis and the parabola $y = 9 - x^2$.
5. Find $f(x)$ if $f''(x) = x$ and $f(0) = 1$ and $f(2) = 3$.

5.2 The definite integral

Example

The velocity of a baseball moving straight up and down under the acceleration of gravity is $v(t) = -32t + 48$, where time t is given in seconds and velocity v is in ft/s. When $t = 0$, the baseball had a height of 0 ft.

1. What was the initial velocity of the baseball?
2. What was the maximum height of the baseball?
3. What was the height of the baseball at time $t = 2$?