Definition of the Definite Integral

Definition

Given a closed interval $[a, b]$. A *partition* of $[a, b]$ is a set of numbers $\mathcal{P} = \{x_0, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$ that divides $[a, b]$ into n subintervals. Suppose that all subintervals are of equal length, $\Delta x = (b - a)/n$. In each subinterval $[x_{k-1}, x_k]$, pick a number x_k^*.
A function f is *integrable* on $[a, b]$ if

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

exists independent of the choices of x_k^* in $[x_{k-1}, x_k]$. In this case, the *definite integral of f from a to b*, written

$$\int_a^b f(x)\, dx$$

is the value of that limit.

5.4 The Fundamental Theorem of Calculus

The mean value theorem of integration

Theorem

Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b - a)$$

Definition

The **average value of f on** $[a, b]$ is $f(c)$, where c is the value guaranteed by the mean value theorem of integration. That is,

$$\text{Average value of } f \text{ on } [a, b] = f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx$$

Theorem
Part 1 of the FTC

Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt$. Then F is a differentiable function on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Remark

1. $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f.
2. The rate at which area is being added under the graph of $y = f(x)$ at any point x is equal to the value of the function $f(x)$.

Example

1. $\frac{d}{dx}\int_3^x \sin t^2\,dt =$

   = ______________________________

2. $\frac{d}{dx}\int_x^7 e^{t^3}\,dt =$

   = ______________________________

3. $\frac{d}{dx}\int_{-1}^{x^2} \ln t\,dt =$

   = ______________________________

4. $\frac{d}{dx}\int_x^{x^2} t^2 + 3t\,dt =$

   = ______________________________

5.4 The Fundamental Theorem of Calculus

Theorem
Part 2 of the FTC

Let f be continuous on $[a, b]$, and let F be any antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example

1. $\int_0^4 4x - x^2\,dx =$ ____________
2. $\int_0^{\pi/3} \sin x\,dx =$ ____________
3. $\int_0^2 3^x\,dx =$ ________________
4. $\int_0^2 x(x-1)(x-2)\,dx =$ ____

5.4 The Fundamental Theorem of Calculus

Motion

Remark

Recall that $\frac{d}{dt}(s) = v$ and that $\frac{d}{dt}(v) = a$. So by the FTC

$$\int_a^b v(t)\,dt = s(t)\Big|_a^b = s(b) - s(a)$$

and

$$\int_a^b a(t)\,dt = v(t)\Big|_a^b = v(b) - v(a)$$

Example

Suppose an object moves in a straight line with velocity $v(t) = t^2 - 2t$ m/s.

1. When is the object moving forward? backward?
2. What is the object's displacement over $[1, 3]$?
3. What is the distance the object travels over $[1, 3]$?

5.4 The Fundamental Theorem of Calculus

Area between curves

Theorem

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$, where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$ is

$$\int_a^b f(x) - g(x)\, dx$$

Example

1. Find the area enclosed by $y = x^2 - 2x + 5$ and $y = 5x - 5$.
2. Find the area of one lobe bounded by the curves $y = \sin x$ and $y = \cos x$.

5.4 The Fundamental Theorem of Calculus

The mean value theorem of integration

Theorem

Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b - a)$$

Definition

The **average value of f on** $[a, b]$ is $f(c)$, where c is the value guaranteed by the mean value theorem of integration. That is,

$$\text{Average value of } f \text{ on } [a, b] = f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx$$

Example

An object moves along a straight line with velocity $v(t) = t^2 - 2t$ m/s. What is its average velocity over $[0, 3]$?

5.4 The Fundamental Theorem of Calculus

Just checking. . . .

1. Find $\dfrac{d}{dx}\displaystyle\int_3^{x^3} \sin t^2\, dt$.
2. Find the area between $y = x$ and $y = x^2/4$.
3. Find a value c guaranteed by the mean value theorem for integration for $\displaystyle\int_0^2 x^2\, dx$.
4. Find $\displaystyle\int_1^e \frac{dx}{x}$.
5. Find the area bounded by $y = 3x^2 - 3$ and the x-axis over the interval $[-2, 2]$.

5.5 Numerical integration

Example

Estimate $\displaystyle\int_0^1 e^{-x^2}\, dx$ using 5 equally spaced subintervals and

1. the left-endpoint method
2. the right-endpoint method
3. trapezoids

x_i	$f(x_i) = e^{-x_i^2}$
0	1
0.2	0.9608
0.4	0.8521
0.6	0.6977
0.8	0.5273
1	0.3679