

Example

- ① Find $\frac{d}{dx} (x^2 + 3x - 5)^{10}$.
- ② Find $\int (20x + 30)(x^2 + 3x - 5)^9 dx$.

Remark

More generally $\frac{d}{dx} (F(g(x))) = F'(g(x))g'(x)$ and so

$$\int F'(g(x))g'(x) dx = F(g(x)) + C.$$

Letting $u = g(x)$, so that $du = g'(x) dx$, we have

$$\int F'(u) du = F(u) + C = F(g(x)) + C.$$

Theorem

Let F and g be differentiable functions, where the range of g is contained in the domain of F . Then

$$\int F'(g(x))g'(x) dx = F(g(x)) + C$$

If $u = g(x)$ so that $du = g'(x) dx$, then

$$\int F'(g(x))g'(x) dx = \int F'(u) du = F(u) + C = F(g(x)) + C$$

Example

$$\textcircled{1} \int x e^{x^2} dx = \underline{\hspace{10cm}}$$

$$\textcircled{2} \int x^2 \cos x^3 dx = \underline{\hspace{10cm}}$$

$$\textcircled{3} \int \sin x \cos x dx = \underline{\hspace{10cm}}$$

$$\textcircled{4} \int \frac{e^x + 1}{e^x} dx = \underline{\hspace{10cm}}$$

$$\textcircled{5} \int \frac{3x^2 - 5x + 7}{x + 1} dx = \underline{\hspace{10cm}}$$

$$\textcircled{6} \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = \underline{\hspace{10cm}}$$

Example

$$\textcircled{1} \int \frac{x^3 - x}{\sqrt{x}} dx = \underline{\hspace{10cm}}$$

$$\textcircled{2} \int \frac{\ln x}{x} dx = \underline{\hspace{10cm}}$$

$$\textcircled{3} \int \frac{\ln \sqrt{x}}{x} dx = \underline{\hspace{10cm}}$$

$$\textcircled{4} \int \tan x dx = \underline{\hspace{10cm}}$$

$$\textcircled{5} \int x \sqrt{1 - x^2} dx = \underline{\hspace{10cm}}$$

$$\textcircled{6} \int x \sqrt{1 - x} dx = \underline{\hspace{10cm}}$$

Trigonometric integrals

Theorem

$$\textcircled{1} \int \cos x \, dx = \sin x + C$$

$$\textcircled{2} \int \sin x \, dx = -\cos x + C$$

$$\textcircled{3} \int \tan x \, dx = -\ln |\cos x| + C$$

$$\textcircled{4} \int \cot x \, dx = \ln |\sin x| + C$$

$$\textcircled{5} \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\textcircled{6} \int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

Trigonometric integrals

Theorem

Angle Addition Formulas

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

Example

$$\textcircled{1} \sin 2\theta = \underline{\hspace{2cm}} \qquad \textcircled{2} \cos 2\theta = \underline{\hspace{2cm}}$$

Example

$$\textcircled{1} \sin^2(\theta/2) = \underline{\hspace{2cm}} \qquad \textcircled{2} \cos^2(\theta/2) = \underline{\hspace{2cm}}$$

Trigonometric integrals

Example

$$\textcircled{1} \int \cos^2 x \, dx = \underline{\hspace{10cm}}$$

$$\textcircled{2} \int \sin^2 x \, dx = \underline{\hspace{10cm}}$$

$$\textcircled{3} \int \tan^2 x \, dx = \underline{\hspace{10cm}}$$

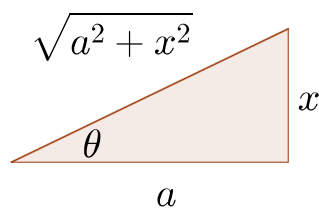
$$\textcircled{4} \int \cos^4 x \, dx = \underline{\hspace{10cm}}$$

$$\textcircled{5} \int \cos^3 x \, dx = \underline{\hspace{10cm}}$$

$$\textcircled{6} \int \tan^3 x \, dx = \underline{\hspace{10cm}}$$

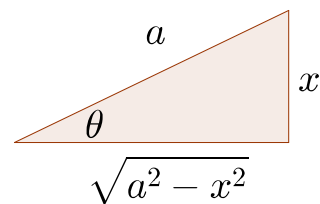
Trigonometric substitution

A sum or difference of squares underneath a square root sign may be interpreted as one side of a right triangle.



$$x = a \tan \theta$$

$$dx = a \sec^2 \theta \, d\theta$$



$$x = a \sin \theta$$

$$dx = a \cos \theta \, d\theta$$

By making an appropriate substitution, we can convert integrals involving sums or differences of squares underneath a square root into trigonometric integrals.

Trigonometric substitution

Example

$$\textcircled{1} \int \frac{dx}{9+x^2} = \underline{\hspace{10cm}}$$

$$\textcircled{2} \int \frac{dx}{\sqrt{4-x^2}} = \underline{\hspace{10cm}}$$

$$\textcircled{3} \int \frac{9-x}{\sqrt{9-x^2}} dx = \underline{\hspace{10cm}}$$

$$\textcircled{4} \int \frac{x^2 dx}{\sqrt{9-x^2}} = \underline{\hspace{10cm}}$$

$$\textcircled{5} \int \frac{1}{x^2-4x+13} dx = \underline{\hspace{10cm}}$$

Substitution in definite integrals

Theorem

Let F and g be differentiable functions, where the range of g is contained in the domain of F . Then

$$\int_a^b F'(g(x))g'(x) dx = \int_{u=g(a)}^{u=g(b)} F'(u) du$$

Remark

So, when evaluating a definite integral where a substitution has been made, one can either

- convert the antiderivative to a function of x and use the original x -limits, or
- leave the antiderivative as a function of u and convert the x -limits to u -limits.

Substitution in definite integrals

Example

$$\textcircled{1} \int_2^7 \sin(2x + 3) dx = \underline{\hspace{10cm}}$$

$$\textcircled{2} \int_0^{\pi/3} \sin^2 x \cos x dx = \underline{\hspace{10cm}}$$

$$\textcircled{3} \int_1^9 \frac{\ln x^2}{x} dx = \underline{\hspace{10cm}}$$

$$\textcircled{4} \int_1^9 \frac{(\ln x)^2}{x} dx = \underline{\hspace{10cm}}$$

$$\textcircled{5} \int_{-1}^1 \frac{1}{1 + x^2} dx = \underline{\hspace{10cm}}$$

$$\textcircled{6} \int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}} dx = \underline{\hspace{10cm}}$$