Definition

(Informal definition)

If we can make the values of a function $f(x)$ get arbitrarily close to a number L by taking x-values that are sufficiently close to c (but not equal to c), then we say the *limit* of $f(x)$ as x approaches c is L, and we write

$$\lim_{x \to c} f(x) = L$$

Remark

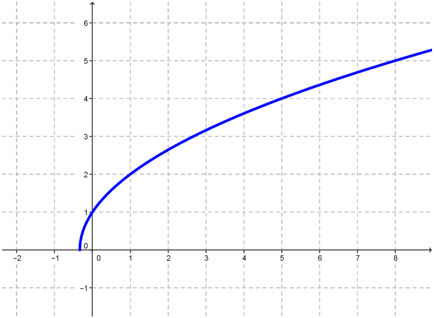
We need to define more clearly what is means to be arbitrarily close and sufficiently close in order to make the informal definition of limit given above more rigorous. We'll do that in the next section. For now, we'll use this informal definition to explore the idea of a limit by estimating limits (a) graphically and (b) numerically.

Estimating limits graphically

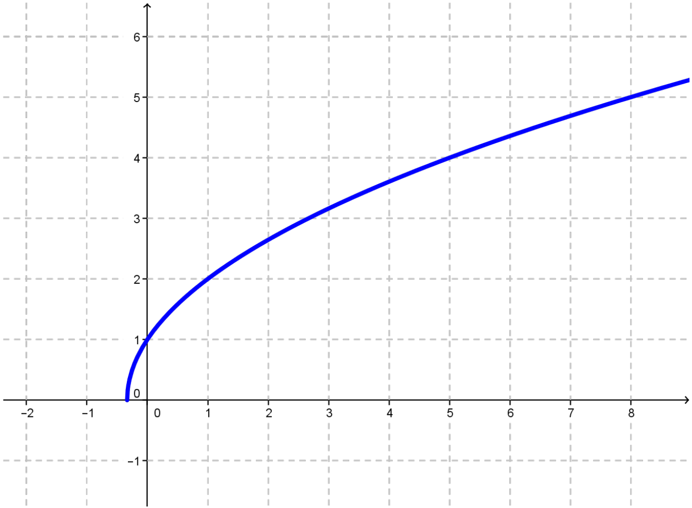
Example

Find

- $\lim_{x\to 5} \sqrt{3x+1} =$ ____________
- $\lim_{x\to 1} \sqrt{3x+1} =$ ____________
- $\lim_{x\to 0} \sqrt{3x+1} =$ ____________
- $\lim_{x\to 8} \sqrt{3x+1} =$ ____________
- $\lim_{x\to 2} \sqrt{3x+1} =$ ____________



$y = \sqrt{3x+1}$



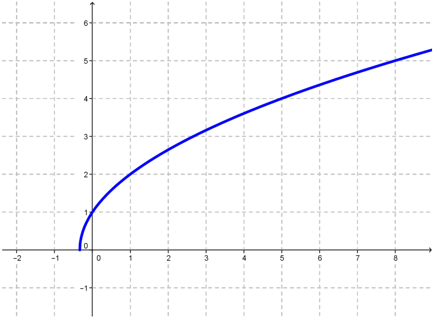
$$y = \sqrt{3x + 1}$$

Estimating limits graphically

Example

Find

- $\lim_{x \to 5} \sqrt{3x+1} = 4$
- $\lim_{x \to 1} \sqrt{3x+1} = 2$
- $\lim_{x \to 0} \sqrt{3x+1} = 1$
- $\lim_{x \to 8} \sqrt{3x+1} = 5$
- $\lim_{x \to 2} \sqrt{3x+1} \approx 2.7$



$y = \sqrt{3x+1}$

Estimating limits numerically

Example

Find $\lim_{x \to 3} \dfrac{2x^2 - 2x - 12}{x - 3}$

Solution

Pick x-values near 3 (but not equal to 3) and compute the value of the function.

x	$\frac{2x^2-2x-12}{x-3}$	x	$\frac{2x^2-2x-12}{x-3}$
2.9		3.1	
2.99		3.01	
2.999		3.001	

How close to 3 would x have to be in order to get within

- 0.0002 of the limit?
- 0.1 of the limit?

Estimating limits numerically

Example

Find $\lim_{x \to 3} \dfrac{2x^2 - 2x - 12}{x - 3}$

Solution

Pick x-values near 3 (but not equal to 3) and compute the value of the function.

x	$\frac{2x^2-2x-12}{x-3}$	x	$\frac{2x^2-2x-12}{x-3}$
2.9	9.8	3.1	10.2
2.99	9.98	3.01	10.02
2.999	9.998	3.001	10.002

How close to 3 would x have to be in order to get within

- 0.0002 of the limit? (within 0.0001 of 3)
- 0.1 of the limit? (within 0.05 of 3)

When limits do not exist

Remark

A function may not have a limit for all values of x. There are three ways a limit may fail to exist as x approaches c:

1. The function may approach different values on either side of c or fail to exist near c.
2. The function may grow without bound (i.e. tend toward $+\infty$ or $-\infty$) as x approaches c.
3. The function may oscillate infinitely often as x approaches c without staying near any single value.

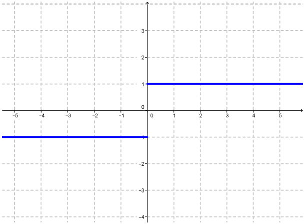
In the following slides we give examples of each case.

When limits do not exist: (1) different values from left and right

Example

Consider

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & x > 0 \\ \frac{-x}{x} = -1 & x < 0 \end{cases}$$



$y = |x|/x$

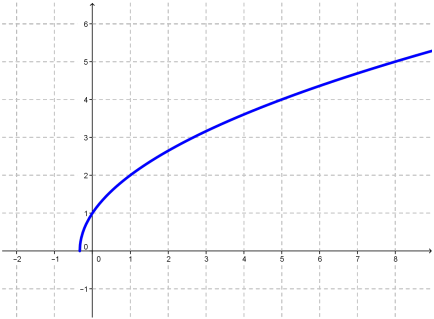
$\lim_{x\to 0} \frac{|x|}{x}$ does not exist because $\lim_{x\to 0^+} \frac{|x|}{x} = 1$ (the right-hand limit) while $\lim_{x\to 0^-} \frac{|x|}{x} = -1$ (the left-hand limit).

When limits do not exist: (1) function fails to exist near c

Example

Consider

$f(x) = \sqrt{3x+1}$



$y = \sqrt{3x+1}$

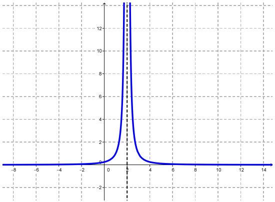
$\lim_{x \to -1/3} \sqrt{3x+1}$ does not exist because $f(x) = \sqrt{3x+1}$ does not exist for $x < -1/3$.

When limits do not exist: (2) function grows without bound

Example

Consider

$$f(x) = \frac{1}{(x-2)^2}$$



$y = 1/(x-2)^2$

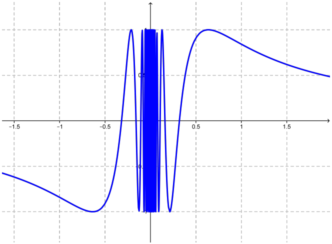
$\lim_{x\to 2} 1/(x-2)^2$ does not exist because there is no number L that the function $1/(x-2)^2$ gets closer to as x approaches 2; indeed, the values of the function $1/(x-2)^2$ can be made as large as we'd like by taking x sufficiently close to 2. In other words, the function grows without bound as x approaches 2.

When limits do not exist: (3) function oscillates without staying near a single value

Example

Consider

$f(x) = \sin(1/x)$



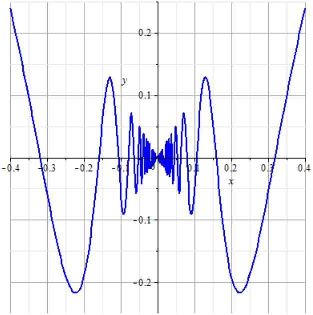
$y = \sin(1/x)$

$\lim_{x \to 0} \sin(1/x)$ does not exist because there is no number L that the function $\sin(1/x)$ gets closer to as x approaches 0; indeed, the values of the function $\sin(1/x)$ oscillate more rapidly between -1 and 1 as x approaches 0.

Example

On the other hand, consider

$$f(x) = x\sin(1/x)$$



$y = x\sin(1/x)$

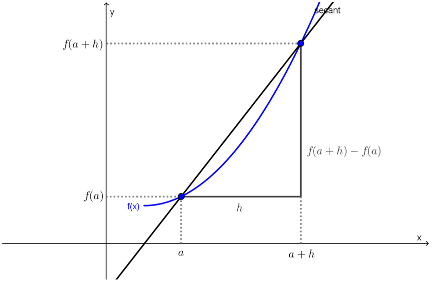
$\lim_{x \to 0} x\sin(1/x)$ does exist because, even though the function oscillates infinitely often near $x = 0$, the function values are getting closer and closer to zero.

Limits of difference quotients

Remark

A *difference quotient* computes the slope of a secant line between two points $(a, f(a))$ and $(a+h, f(a+h))$:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

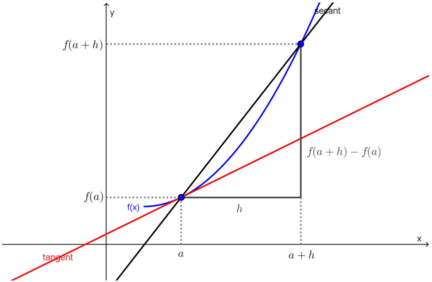


Limits of difference quotients

Remark

Taking the limit of these difference quotients as $h \to 0$ will give us the slope of the line that is tangent to $y = f(x)$ at the point $(a, f(a))$:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Link: GeoGebra tangent_line_slope.ggb

Just checking. . . .

1. True or false: The limit of $f(x)$ as x approaches 5 is $f(5)$.
2. Approximate $\lim_{x\to 0} f(x)$ numerically and graphically, where $f(x) = \left\{ \begin{array}{ll} \cos x & x \leq 0 \\ x^2 + 3x + 1 & x > 0 \end{array} \right\}$, or state why the limit does not exist.
3. Approximate $\lim_{x\to 2} \dfrac{x^2 + 7x + 10}{x^2 - 4x + 4}$ numerically and graphically, or state why the limit does not exist.
4. Approximate $\lim_{x\to 2} f(x)$ numerically and graphically, where $f(x) = \left\{ \begin{array}{ll} x + 1 & x < 2 \\ 3x - 5 & x \geq 2 \end{array} \right\}$, or state why the limit does not exist.
5. Consider the function $f(x) = \ln x$ and the point $a = 5$. Approximate the limit of the difference quotient $\dfrac{f(a + h) - f(a)}{h}$ by using $h = \pm 0.1, \pm 0.01$.

Toward a more rigorous definition

Definition

(Informal definitions) Given a function $y = f(x)$ and an x-value c, we say that $\lim_{x \to c} f(x) = L$ provided

1. $y = f(x)$ is near L whenever x is near c.
2. whenever x is within a certain tolerance level of c, then the corresponding value $y = f(x)$ is within a certain tolerance level of L.

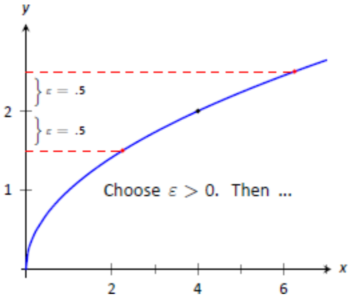
Remark

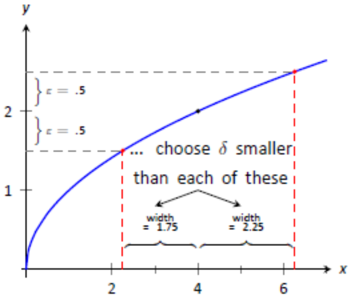
The tolerances for x and y are different. The y-tolerance is called ε, and the x-tolerance is called δ.

A rigorous definition of limit

Definition

(Rigorous definition) Let f be a function defined on an open interval containing c. Then $\lim_{x \to c} f(x) = L$ provided that given any $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.





With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Examples

Example

1. Show $\lim_{x \to 9} \sqrt{x} = 3$. (Cf. §1.2: Example 6)
2. Show $\lim_{x \to 4} x^2 = 16$. (Cf. §1.2: Example 7)
3. Show $\lim_{x \to 0} e^x = 1$. (Cf. §1.2: Example 8)
4. Show $\lim_{x \to c} e^x = e^c$.

Remark

This last example show that the function $f(x) = e^x$ is *continuous* at all values of x. More generally, a function $f(x)$ is continuous at c if $\lim_{x \to c} f(x) = f(c)$. We'll explore this important idea of continuity more in §1.5.

Just checking. . . .

1. What's wrong with the following "definition" of a limit?

   The limit of $f(x)$ as x approaches a is K means that given any $\delta > 0$, there exists an $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$ we have $0 < |x - c| < \delta$.

2. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 3} 5 = 5$.
3. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 2} 3 - 2x = -1$.
4. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 3} x^2 - 3 = 6$.
5. Using an $\varepsilon - \delta$ argument, show $\lim_{x\to 0} e^{2x} - 1 = 0$.

Limit Laws

Theorem

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \quad \lim_{x\to c} g(x) = K$$

Then the following limits hold:

1. $\lim_{x\to c} b = b$
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
4. $\lim_{x\to c} b \cdot f(x) = bL$
5. $\lim_{x\to c} f(x) \cdot g(x) = LK$
6. $\lim_{x\to c} f(x)/g(x) = L/K (K \neq 0)$
7. $\lim_{x\to c} [f(x)]^n = L^n$
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. And if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to L} g(x) = K$, then $\lim_{x\to c} g(f(x)) = K$.

Using Limit Laws

Example

Suppose $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = -2$ and $p(x) = x^2 - 3x + 1$. Find:

1. $\lim_{x\to 2} f(x) + g(x) =$ ____________
2. $\lim_{x\to 2} [g(x)]^3 =$ ________________
3. $\lim_{x\to 2} p(x) =$ ___________________
4. $\lim_{x\to 2} f(x) - 3p(x) =$ __________

Using Limit Laws

Example

Suppose $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = -2$ and $p(x) = x^2 - 3x + 1$. Find:

1. $\lim_{x\to 2} f(x) + g(x) = 1$
2. $\lim_{x\to 2} [g(x)]^3 = -8$
3. $\lim_{x\to 2} p(x) = -1$
4. $\lim_{x\to 2} f(x) - 3p(x) = 6$

Remark

Notice that

$$\lim_{x\to 2} p(x) = -1 = p(2)$$

Whenever $\lim_{x\to c} f(x) = f(c)$, we say that f is *continuous at c*. We will study continuity in greater detail in §1.5.

Other Limit Laws

Theorem

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x \to c} p(x) = p(c)$
2. $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, where $q(c) \neq 0$

Remark

So polynomials are continuous everywhere and rational functions are continuous at every point in their domain. The next theorem lists other common continuous functions.

Other Limit Laws

Theorem

Let c be a real number in the domain of the given function, let n be a positive integer, and let $a > 0$. Then the following limits hold:

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \csc x = \csc c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to c} a^x = a^c$
8. $\lim_{x\to c} \ln x = \ln c$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example

Evaluate the following limits:

1. $\lim_{x\to \pi/3} \cos x =$ ________________
2. $\lim_{x\to 3} \csc^2 x - \cot^2 x =$ ________
3. $\lim_{x\to 0} \ln e^{3x} =$ ________________
4. $\lim_{x\to 1} e^{\sqrt{4x}} =$ ________________

Other Limit Laws

Theorem

Let c be a real number in the domain of the given function, let n be a positive integer, and let $a > 0$. Then the following limits hold:

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \csc x = \csc c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to c} a^x = a^c$
8. $\lim_{x\to c} \ln x = \ln c$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example

Evaluate the following limits:

1. $\lim_{x\to\pi/3} \cos x = 1/2$
2. $\lim_{x\to 3} \csc^2 x - \cot^2 x = 1$
3. $\lim_{x\to 0} \ln e^{3x} = 0$
4. $\lim_{x\to 1} e^{\sqrt{4x}} = e^2$

Squeeze theorem

Theorem

(Squeeze theorem) Let f, g and h be functions on an open interval I containing c such that for all x in I we have $f(x) \leq g(x) \leq h(x)$. If

$$\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$$

then

$$\lim_{x \to c} g(x) = L$$

Example

(A fundamental trig limit) $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

Remark

This says that $\sin x$ and x are approaching 0 at the same rate.

Special limits

Theorem

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
3. $\lim_{x \to 0} (1 + x)^{1/x} = e$
4. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Remark

So, $\cos x - 1$ approaches 0 faster than x does, while $e^x - 1$ and x approach 0 at the same rate. These are all examples of *indeterminate forms*, which we will study in more detail in a later section (§6.7).

Special limits

Example

Find the following limits:

1. $\lim_{x \to 0} \frac{\sin(3x)}{x} = \underline{\qquad\qquad}$
2. $\lim_{x \to 0} \frac{\sin(5x)}{8x} = \underline{\qquad\qquad}$
3. $\lim_{x \to 0} \frac{\tan(4x)}{3x} = \underline{\qquad\qquad}$
4. $\lim_{x \to 0} x \csc(2x) = \underline{\qquad\qquad}$

Remark

If numerator and denominator each contain a factor that is making the denominator zero, canceling the common factor can make it possible to evaluate the limit.

Example

Find $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$.

Special limits

Example

Find the following limits:

1. $\lim_{x\to 0} \frac{\sin(3x)}{x} = 3$
2. $\lim_{x\to 0} \frac{\sin(5x)}{8x} = 5/8$
3. $\lim_{x\to 0} \frac{\tan(4x)}{3x} = 4/3$
4. $\lim_{x\to 0} x\csc(2x) = 1/2$

Remark

If numerator and denominator each contain a factor that is making the denominator zero, canceling the common factor can make it possible to evaluate the limit.

Example

Find $\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$.

Using algebra to find limits

Theorem

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x\to c} g(x) = L$. Then

$$\lim_{x\to c} f(x) = L$$

Example

Find the following limits:

1. $\lim_{x\to 0} \dfrac{x^2+3x}{x^2-2x} =$ ____________
2. $\lim_{x\to -1} \dfrac{x^2+9x+8}{x^2-6x-7} =$ ________
3. $\lim_{x\to 6} \dfrac{x^2-4x-12}{x^2-13x+42} =$ _______
4. $\lim_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2} =$ ______________

Using algebra to find limits

Theorem

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x \to c} g(x) = L$. Then

$$\lim_{x \to c} f(x) = L$$

Example

Find the following limits:

1. $\lim_{x \to 0} \frac{x^2 + 3x}{x^2 - 2x} = -3/2$
2. $\lim_{x \to -1} \frac{x^2 + 9x + 8}{x^2 - 6x - 7} = -7/8$
3. $\lim_{x \to 6} \frac{x^2 - 4x - 12}{x^2 - 13x + 42} = -8$
4. $\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = -1/4$

Just checking. . . .

1. You are given the following information:
   - $\lim_{x\to 1} f(x) = 0 = \lim_{x\to 1} g(x)$
   - $\lim_{x\to 1} f(x)/g(x) = 2$.

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?
2. Find $\lim_{x\to 0} \dfrac{5x}{\cos(3x)}$.
3. Find $\lim_{x\to 0} \dfrac{5x}{\sin(3x)}$.
4. Find $\lim_{x\to \pi/4} \cos x \sin x$.
5. Find $\lim_{x\to -2} \dfrac{x^2 - 5x - 14}{x^2 + 10x + 16}$.

Left- and right-hand limits

Definition

Let f be a function defined on an open interval containing c. The notation

$$\lim_{x \to c^-} f(x) = L,$$

which is read as "the limit of $f(x)$ as x approaches c from the left is L," or "the left-hand limit of f at c is L," means that:

> given any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < c - x < \delta$, we have $|f(x) - L| < \varepsilon$.

Remark

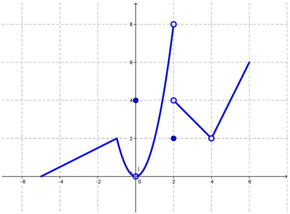
What changes need to be made to define a right-hand limit at c?

Left- and right-hand limits

Example

The graph of the function $f(x)$ is shown below at right. Based on the graph, determine the following:

1. $\lim_{x\to -1^-} f(x) =$ ________________
2. $\lim_{x\to -1^+} f(x) =$ ________________
3. $\lim_{x\to -1} f(x) =$ ________________
4. $f(-1) =$ ________________
5. $\lim_{x\to 2^-} f(x) =$ ________________
6. $\lim_{x\to 2^+} f(x) =$ ________________
7. $\lim_{x\to -2} f(x) =$ ________________
8. $f(2) =$ ________________
9. $\lim_{x\to 0} f(x) =$ ________________
10. $f(0) =$ ________________
11. $\lim_{x\to 4} f(x) =$ ________________
12. $f(4) =$ ________________

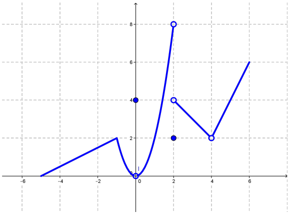


Left- and right-hand limits

Example

The graph of the function $f(x)$ is shown below at right. Based on the graph, determine the following:

1. $\lim_{x\to -1^-} f(x) = 2$
2. $\lim_{x\to -1^+} f(x) = 2$
3. $\lim_{x\to -1} f(x) = 2$
4. $f(-1) = 2$
5. $\lim_{x\to 2^-} f(x) = 8$
6. $\lim_{x\to 2^+} f(x) = 4$
7. $\lim_{x\to -2} f(x) =$ dne
8. $f(2) = 2$
9. $\lim_{x\to 0} f(x) = 0$
10. $f(0) = 4$
11. $\lim_{x\to 4} f(x) = 2$
12. $f(4) =$ dne



Left- and right-hand limits

Remark

From this example, we learn two important things about limits. First, $\lim_{x\to c} f(x)$ and $f(c)$ are independent of one another:

> $\lim_{x\to c} f(x)$ can be $f(c)$, but it need not be; in fact, $\lim_{x\to c} f(x)$ can exist even if $f(c)$ doesn't exist, and vice versa.

The second important thing we learn from this example is . . .

Theorem

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L$$

Just checking. . . .

1. True or false: if $\lim_{x\to 5^-} f(x) = 3$, then $\lim_{x\to 5} f(x) = 3$.
2. True or false: if $\lim_{x\to 5} f(x) = 3$, then $\lim_{x\to 5^-} f(x) = 3$.
3. True or false: if $\lim_{x\to 5} f(x) = 3$, then $f(5) = 3$.
4. Estimate the limit numerically: $\lim_{x\to 0.2} \dfrac{x^2 + 5.8x - 1.2}{x^2 - 4.2x + 0.8}$
5. Evaluate the limit: $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x^3 - 3x}$

Definition

Let f be a function defined on an open interval I containing c.

1. f is *continuous at c* if $\lim_{x \to c} f(x) = f(c)$.
2. f is continuous on I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is *continuous everywhere*.

Definition

Let f be defined on the closed interval $[a, b]$ for some real numbers a, b. f is *continuous on* $[a, b]$ if:

1. f is continuous on (a, b),
2. $\lim_{x \to a^+} f(x) = f(a)$, and
3. $\lim_{x \to b^-} f(x) = f(b)$.

Example

Determine the domain of the given function and the interval(s) on which it is continuous.

Function	Domain	Interval(s) of continuity
$f(x) = 1/(x-2)$		
$f(x) = \sqrt{x-1}$		
$f(x) = \sqrt{4-x^2}$		
$f(x) = \lvert x \rvert / x$		

Example

Determine the domain of the given function and the interval(s) on which it is continuous.

Function	Domain	Interval(s) of continuity
$f(x) = 1/(x-2)$	$(-\infty, 2) \cup (2, \infty)$	$(-\infty, 2) \cup (2, \infty)$
$f(x) = \sqrt{x-1}$	$[1, \infty)$	$[1, \infty)$
$f(x) = \sqrt{4-x^2}$	$[-2, 2]$	$[-2, 2]$
$f(x) = \lvert x \rvert / x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Properties of continuous functions

Theorem

Let f and g be continuous functions on an interval I, let c be a real number, and let n be a positive integer. The following functions are continuous on I.

1. $f \pm g$
2. $c \cdot f$
3. $f \cdot g$
4. f/g ($g \neq 0$ on I)
5. f^n
6. $\sqrt[n]{f}$
7. If the range of f is J and g is continuous on J, then $g \circ f$ is continuous on I.

Common continuous functions

Theorem

The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \csc x$
5. $f(x) = \sec x$
6. $f(x) = \cot x$
7. $f(x) = \ln x$
8. $f(x) = a^x$ $(a > 0)$
9. $f(x) = \sqrt[n]{x}$ (n a positive integer)

Example

Find the interval(s) on which the following functions are continuous.

1. $f(x) = \sqrt{3 - x} + \sqrt{2x - 1}$ is continuous on ____________________
2. $f(x) = \sqrt{\ln(2x + 3)}$ is continuous on ____________________

Common continuous functions

Theorem

The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
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5. $f(x) = \sec x$
6. $f(x) = \cot x$
7. $f(x) = \ln x$
8. $f(x) = a^x$ $(a > 0)$
9. $f(x) = \sqrt[n]{x}$ (n a positive integer)

Example

Find the interval(s) on which the following functions are continuous.

1. $f(x) = \sqrt{3-x} + \sqrt{2x-1}$ is continuous on $[1/2, 3]$
2. $f(x) = \sqrt{\ln(2x+3)}$ is continuous on $[-1, \infty)$

Intermediate value theorem

Theorem

(Intermediate value theorem) Let f be a continuous function on $[a, b]$. Then for every value y-value in between $f(a)$ and $f(b)$, there is an x-value c in $[a, b]$ such that $f(c) = y$.

Remark

The IVT is an existence theorem: it asserts the existence of an x-value c with $f(c) = y$ for every y in between $f(a)$ and $f(b)$. We can find a good approximation of c using the *bisection method*, as illustrated in the next example.

Example

(Bisection method) Find the root of $f(x) = x^3 + x + 1$ accurate to two decimal places.

Iteration	Interval	Midpoint sign
1	$[-1, 0]$	$f(-0.5) > 0$
2	$[-1, -0.5]$	$f(-0.75) < 0$
3	$[-0.75, -0.5]$	$f(-0.625) > 0$
4	$[-0.75, -0.625]$	$f(-0.6875) < 0$
5	$[-0.6875, -0.625]$	$f(-0.65625) > 0$
6	$[-0.6875, -0.65625]$	$f(-0.67188) > 0$
7	$[-0.6875, -0.67188]$	$f(-0.67969) > 0$
8	$[-0.6875, -0.67969]$	$f(-0.68359) < 0$
9	$[-0.68359, -0.67969]$	

Just checking. . . .

1. True or false: If f is defined on an open interval containing c and $\lim_{x \to c} f(x)$ exists, then f is continuous at c.
2. True or false: If f is continuous at c, then $\lim_{x \to c} f(x)$ exists.
3. True or false: If f is continuous at c, then $\lim_{x \to c} f(x) = f(c)$.
4. True or false: If f is continuous on $[0, 1)$ and on $[1, 2]$, then f is continuous on $[0, 2]$.
5. Let $f(x) = \left\{ \begin{array}{ll} x^2 & x \leq 2 \\ 3 - mx & x > 2 \end{array} \right\}$. Find the value of m that makes f continuous at $x = 2$.

Infinite limits

Definition

We say $\lim_{x\to c} f(x) = \infty$ if for every $M > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) \geq M$

Example

1. $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$
2. $\lim_{x\to \pi/2^-} \tan x = \infty$ and $\lim_{x\to \pi/2^+} \tan x = -\infty$.

Definition

If the limit of $f(x)$ as x approaches c from either the right or the left (or both) is ∞ or $-\infty$, we say that the function has a *vertical asymptote* at c.

Infinite limits

Example

1. $f(x) = \dfrac{x^2}{x^2 - 1}$ has vertical asymptotes at $x = 1$ and $x = -1$.
2. $f(x) = \dfrac{x^3 + 2x^2 - x - 2}{x^2 - 1}$ does not. Indeed, $f(x) = x + 2$ for all x except $x = 1$ and $x = -1$, where is simply has a hole (i.e. point discontinuity).

Remark

The above example illustrates the fact that just because a function has a denominator of zero for particular values of x does *not* mean that it has a vertical asymptote at those values of x. It must also have an infinite limit (from the left, or from the right, or both) in order to have a vertical asymptote.

Limits at infinity

Definition

1. We say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.
2. We say that $\lim_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.
3. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say that $y = L$ is a *horizontal asymptote* of f.

Example

Find the horizontal asymptote(s) of the following functions.

1. $f(x) = \dfrac{x^2}{x^2 - 1}$
2. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$
3. $f(x) = \dfrac{\sin x}{x}$

Limits at infinity

Definition

1. We say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.
2. We say that $\lim_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.
3. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say that $y = L$ is a *horizontal asymptote* of f.

Example

Find the horizontal asymptote(s) of the following functions.

1. $f(x) = \dfrac{x^2}{x^2 - 1}$ $y = 1$
2. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$ $y = \pm 3$
3. $f(x) = \dfrac{\sin x}{x}$ $y = 0$

Limits at infinity

Theorem

Suppose we have a rational function

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

where any of the coefficients may be zero except for a_n and b_m.

1. If $n = m$, then $\lim_{x \to \pm\infty} f(x) = \frac{a_n}{b_m}$
2. If $n < m$, then $\lim_{x \to \pm\infty} f(x) = 0$.
3. If $n > m$, then $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ are both infinite.

Limits at infinity

Example

Find the following limits.

1. $\lim_{x\to\infty} \dfrac{6x - 2x^2}{x^2 - 2x - 3} =$ ________
2. $\lim_{x\to-\infty} \dfrac{6x - 2x^3}{x^2 - 2x - 3} =$ ________
3. $\lim_{x\to\infty} \dfrac{6x - 2x^2}{x^3 - 2x - 3} =$ ________
4. $\lim_{x\to 3} \dfrac{6x - 2x^2}{x^2 - 2x - 3} =$ ________

Limits at infinity

Example

Find the following limits.

1. $\lim_{x\to\infty} \dfrac{6x - 2x^2}{x^2 - 2x - 3} = -2$
2. $\lim_{x\to-\infty} \dfrac{6x - 2x^3}{x^2 - 2x - 3} = \infty$
3. $\lim_{x\to\infty} \dfrac{6x - 2x^2}{x^3 - 2x - 3} = 0$
4. $\lim_{x\to 3} \dfrac{6x - 2x^2}{x^2 - 2x - 3} = -3/2$

Just checking. . . .

1. True or false: If $\lim_{x\to 3} f(x) = \infty$, then f has a vertical asymptote at $x = 3$.
2. True or false: If $\lim_{x\to 3} f(x) = \infty$, then $f(3)$ is not defined at $x = 3$.
3. True or false: If $\lim_{x\to 3} f(x) = \infty$, then f is not continuous at $x = 3$.
4. Using a $\varepsilon - \delta$ argument, show that $\lim_{x\to 1} 3x - 1 = 2$.
5. Evaluate the limit $\displaystyle\lim_{x\to -4} \frac{x^2 - 16}{x^2 - 4x - 32}$.