Rates of change

1. The slope of a secant line is

$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

and represents the *average rate of change* over $[a, b]$.
Letting $b = a + h$, we can express the slope of the secant line as

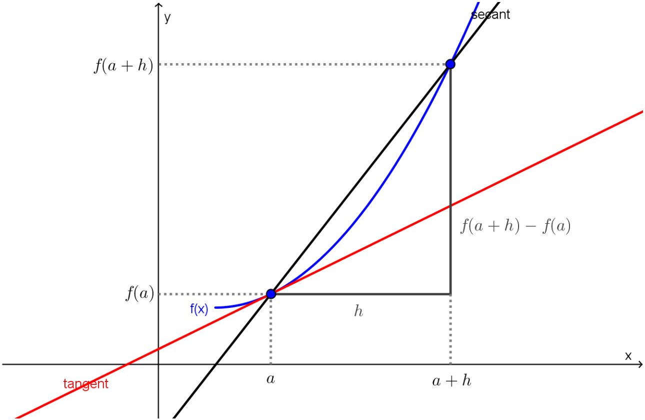
$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$$

2. The slope of the tangent line to $y = f(x)$ at a is the limit of the secant slopes

$$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

and represents the *instantaneous rate of change* at a.

Rates of change



Link: GeoGebra tangent_line_slope.ggb

The derivative at a point

Definition

Let f be a continuous function on an open interval I and let c be in I. The *derivative* of f at c is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided the limit exists.

- If the limit exists we say that f is *differentiable at* c.
- If the limit does not exist, then f is *not differentiable at* c.
- If f is differentiable at every point in I, then f is *differentiable on* I.
- If f is differentiable at c, then $f'(c)$ is the slope of the line that is tangent to $y = f(x)$ at c, and $f'(c)$ represents the instantaneous rate of change in f at c.

Example

Let $f(x) = 2x^2 - 3x + 1$. Find

1. $f'(1) =$ ____________________
   Tangent: ____________________
2. $f'(3) =$ ____________________
   Tangent: ____________________
3. $f'(0) =$ ____________________
   Tangent: ____________________
4. $f'(-2) =$ ____________________
   Tangent: ____________________

and use your answer to write an equation for the tangent line to $y = f(x)$ at the given point.

Remark

An equation for a line of slope m through the point (x_0, y_0) may be written in either

- Point-slope form: $y - y_0 = m(x - x_0)$
- Slope-intercept form: $y = mx + b$

Example

Let $f(x) = 2x^2 - 3x + 1$. Find

1. $f'(1) = 1$

   Tangent: $y = x - 1$
2. $f'(3) = 9$

   Tangent: $y - 10 = 9(x - 3)$
3. $f'(0) = -3$

   Tangent: $y = -3x + 1$
4. $f'(-2) = -11$

   Tangent: $y - 15 = -11(x + 2)$

and use your answer to write an equation for the tangent line to $y = f(x)$ at the given point.

Remark

An equation for a line of slope m through the point (x_0, y_0) may be written in either

- Point-slope form: $y - y_0 = m(x - x_0)$
- Slope-intercept form: $y = mx + b$

Normal lines

Remark

A line with slope m_1 is perpendicular to another line with slope m_2 if and only if $m_1 m_2 = -1$.

Definition

A *normal line* to $y = f(x)$ at c is a line that is perpendicular to the tangent line to $y = f(x)$ at c.

Remark

If $f'(c) \neq 0$, then the slope of the normal line is $-1/f'(c)$. If $f'(c) = 0$, then the normal line is the vertical line through $(c, f(c))$; that is, $x = c$.

Example

Find the normal lines to $f(x) = 2x^2 - 3x + 1$ at

- $x = 1$: ____________________
- $x = 3$: ____________________
- $x = 0$: ____________________
- $x = -2$: ____________________

Normal lines

Remark

A line with slope m_1 is perpendicular to another line with slope m_2 if and only if $m_1 m_2 = -1$.

Definition

A *normal line* to $y = f(x)$ at c is a line that is perpendicular to the tangent line to $y = f(x)$ at c.

Remark

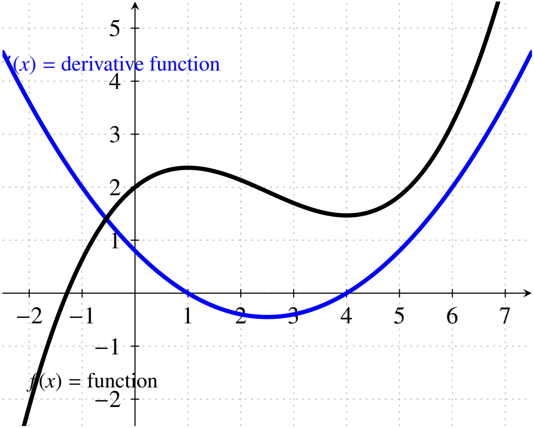
If $f'(c) \neq 0$, then the slope of the normal line is $-1/f'(c)$. If $f'(c) = 0$, then the normal line is the vertical line through $(c, f(c))$; that is, $x = c$.

Example

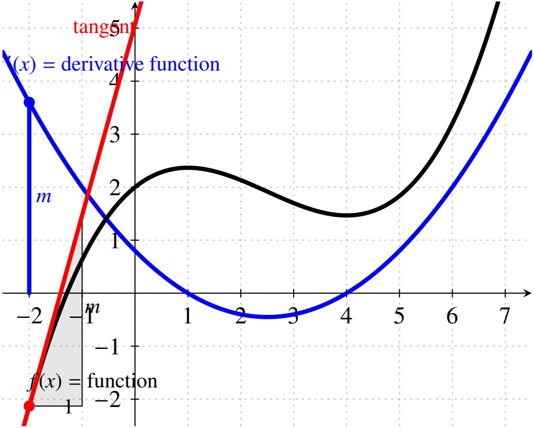
Find the normal lines to $f(x) = 2x^2 - 3x + 1$ at

- $x = 1$: $y = -(x - 1)$
- $x = 3$: $y - 10 = -\frac{1}{9}(x - 3)$
- $x = 0$: $y - 1 = \frac{1}{3}x$
- $x = -2$: $y - 15 = \frac{1}{11}(x + 2)$

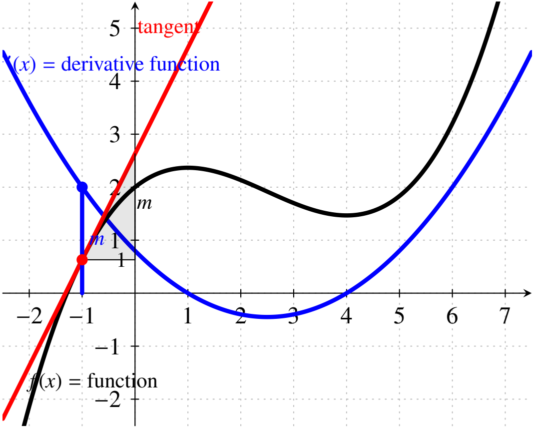
The derivative function



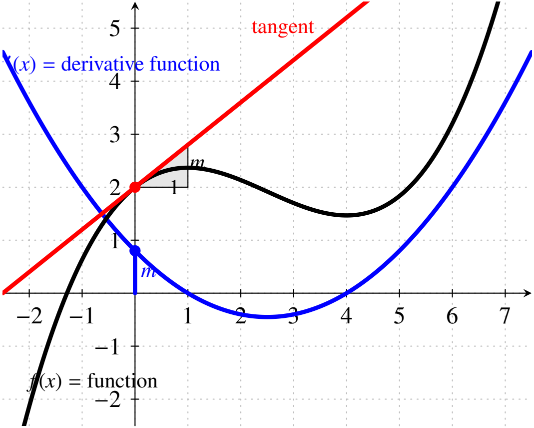
The derivative function



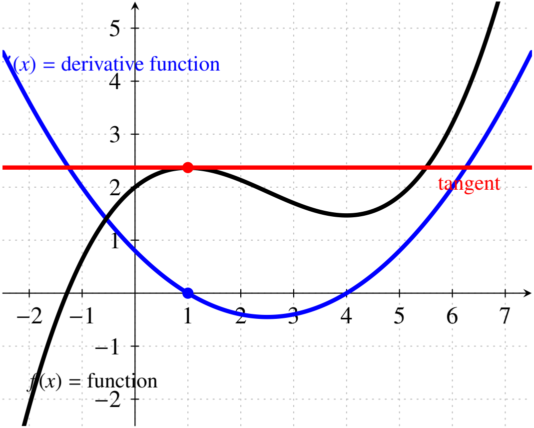
The derivative function



The derivative function



The derivative function



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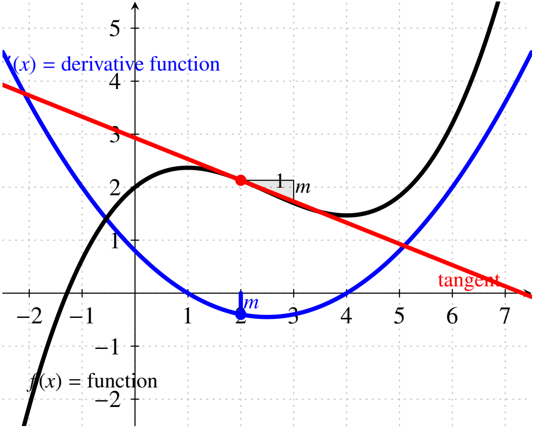
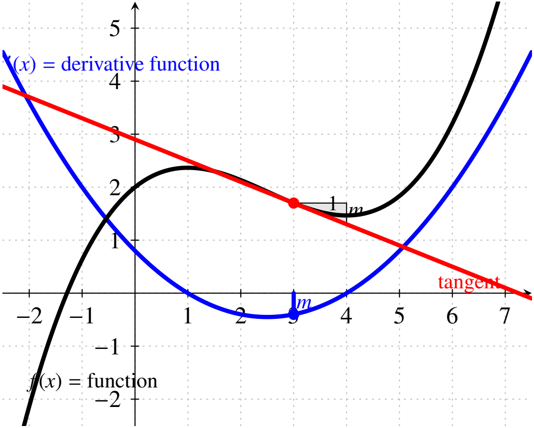
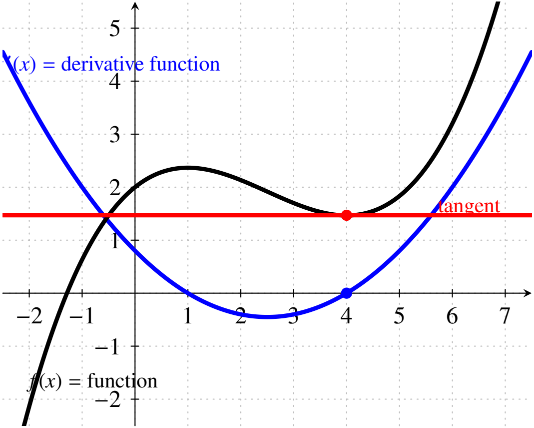


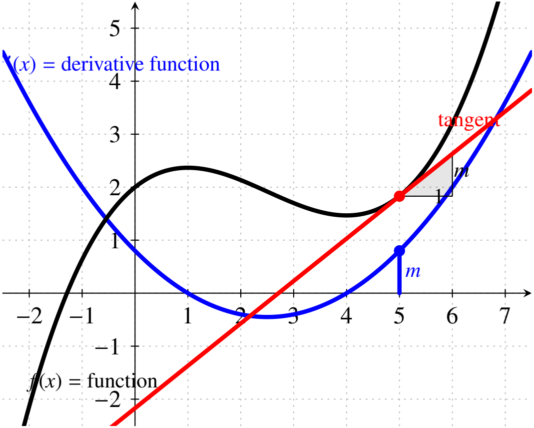
The derivative function



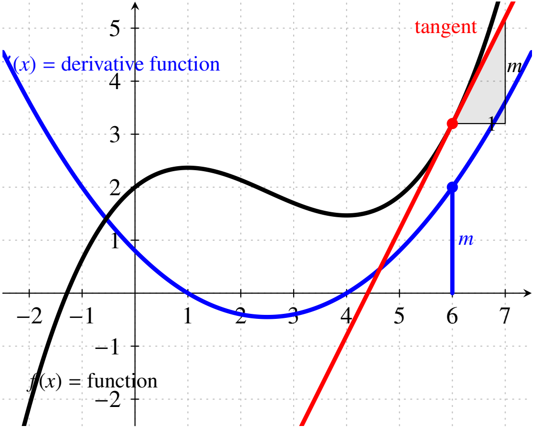
The derivative function



The derivative function



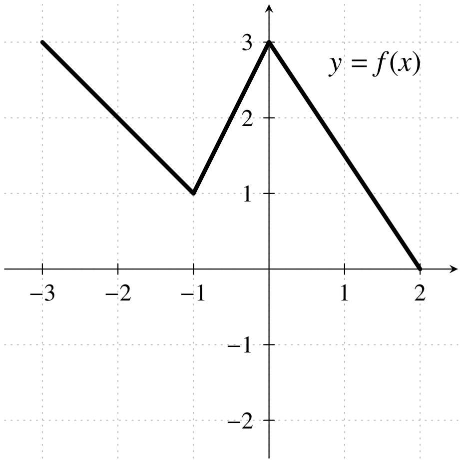
The derivative function



The derivative function

Example

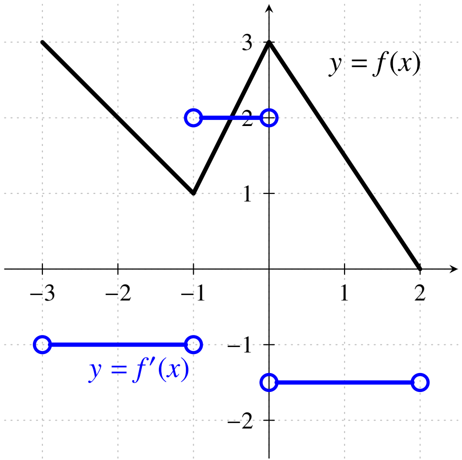
Sketch the derivative of the following function.



The derivative function

Example

Sketch the derivative of the following function.



The derivative function

Definition

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is the *derivative of* f.

If $y = f(x)$, then the following notations all represent the derivative:

$$\underbrace{f'(x) = y' = y'(x)}_{\text{Newton}} = \overbrace{\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f) = \frac{d}{dx}(y)}^{\text{Leibniz}}$$

Remark

In Leibniz notation $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{\Delta y}{\Delta x}$.

The derivative function

Example

Find the derivative of each of the following functions.

1. $f(x) = 2x^2 - 3x + 1$

   $f'(x) =$ ____________________

2. $g(x) = \dfrac{3}{x+2}$

   $g'(x) =$ ____________________

3. $h(x) = 3x - 2$

   $h'(x) =$ ____________________

4. $k(x) = \sqrt{3x}$

   $k'(x) =$ ____________________

Remark

1. Evaluating the derivative function $f'(x)$ at $x = c$ gives us $f'(c)$, which is the slope of the tangent line at c.
2. The line that is tangent to a linear function is the line itself. If $a(x) = |x|$, what is $a'(x)$?

The derivative function

Example

Find the derivative of each of the following functions.

1. $f(x) = 2x^2 - 3x + 1$

   $f'(x) = 4x - 3$

2. $g(x) = \dfrac{3}{x+2}$

   $g'(x) = \dfrac{-3}{(x+2)^2}$

3. $h(x) = 3x - 2$

   $h'(x) = 3$

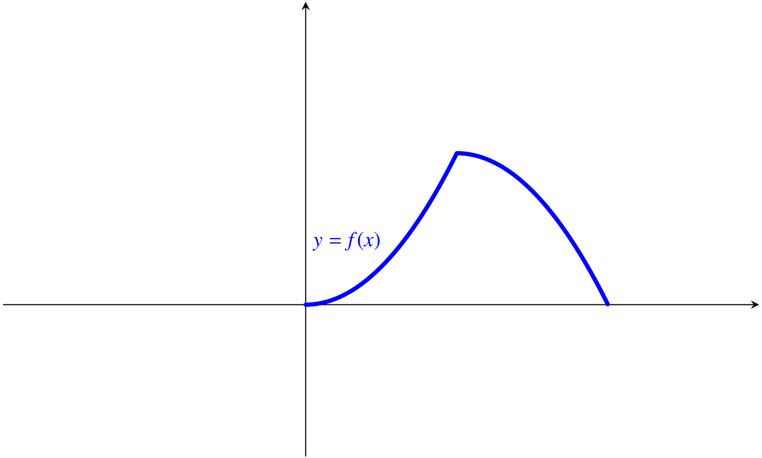
4. $k(x) = \sqrt{3x}$

   $k'(x) = \dfrac{3}{2\sqrt{3x}}$

Remark

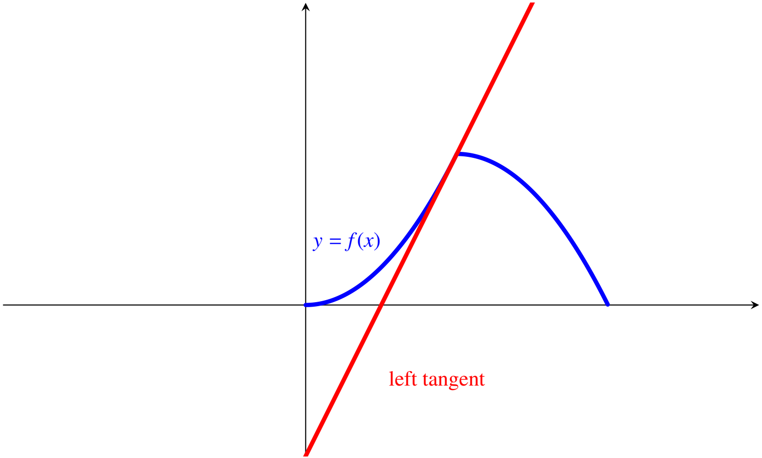
1. Evaluating the derivative function $f'(x)$ at $x = c$ gives us $f'(c)$, which is the slope of the tangent line at c.
2. The line that is tangent to a linear function is the line itself. If $a(x) = |x|$, what is $a'(x)$?

Differentiability



A function is not differentiable at a corner, where the one-sided tangent lines have different slopes.

Differentiability



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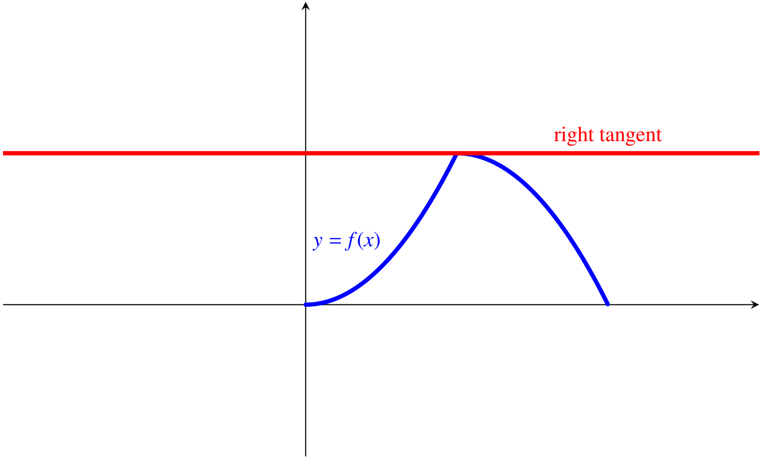

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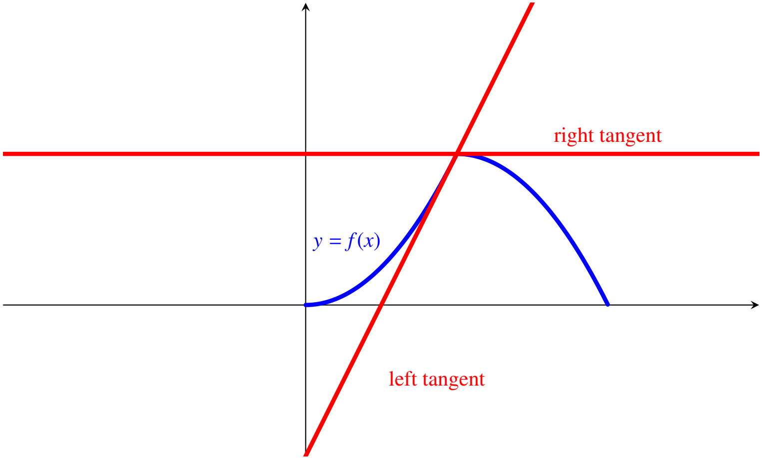

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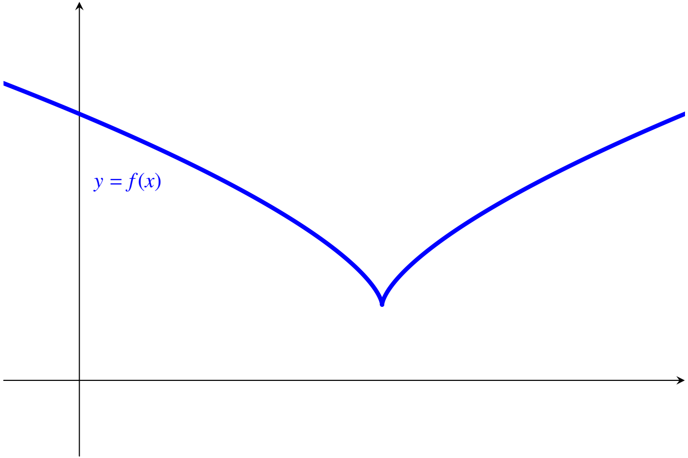
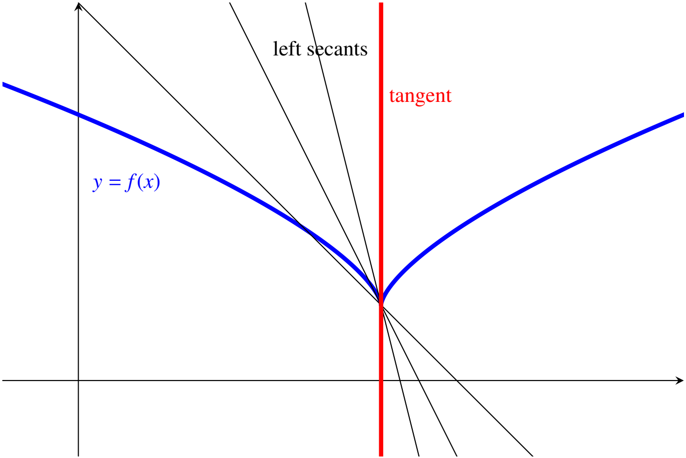


A function is not differentiable at a cusp, where the secant slopes approach $-\infty$ from one side and $+\infty$ from the other.

Differentiability



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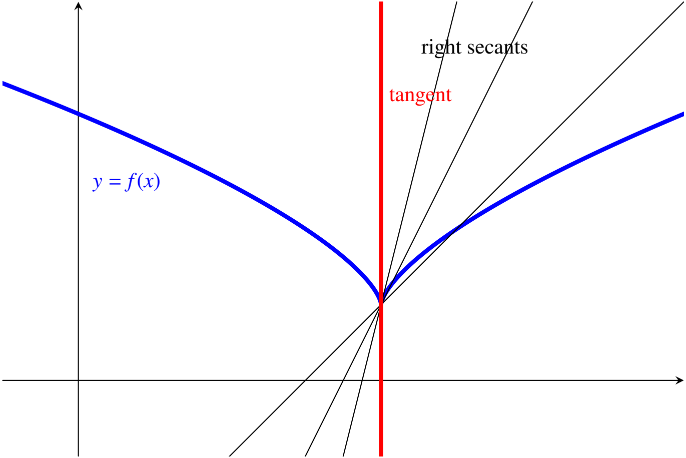

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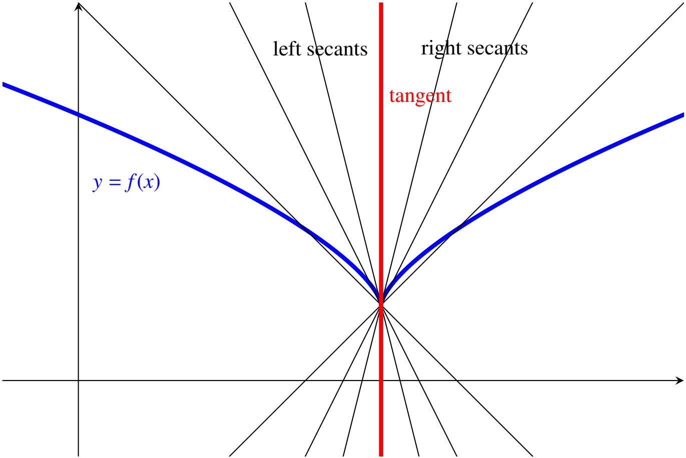

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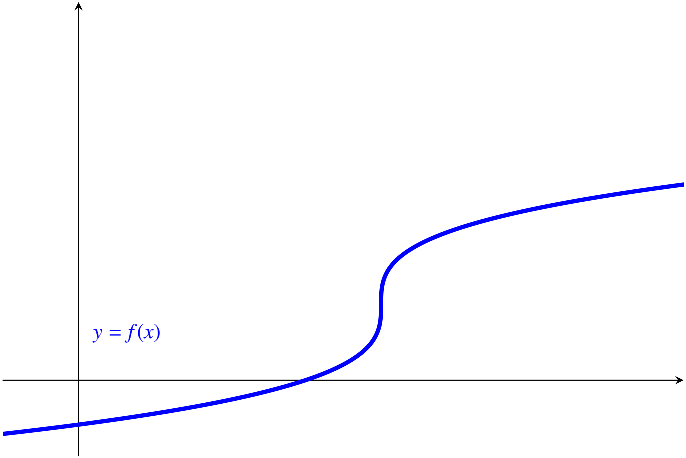
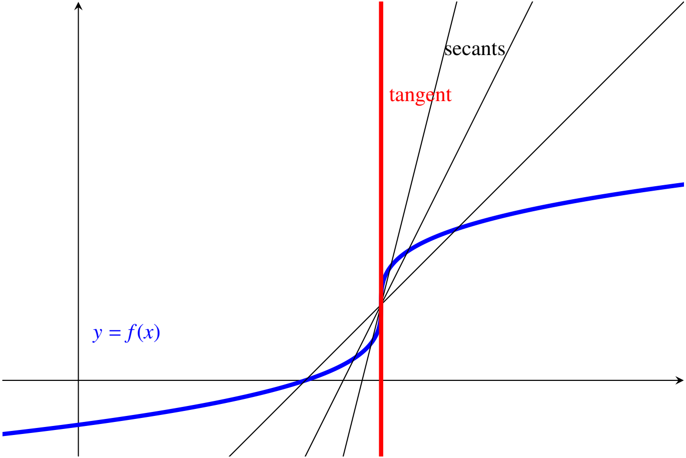


A function is not differentiable at a vertical tangent, where the secant slopes approach $+\infty$ from both sides (or $-\infty$ from both sides).

Differentiability



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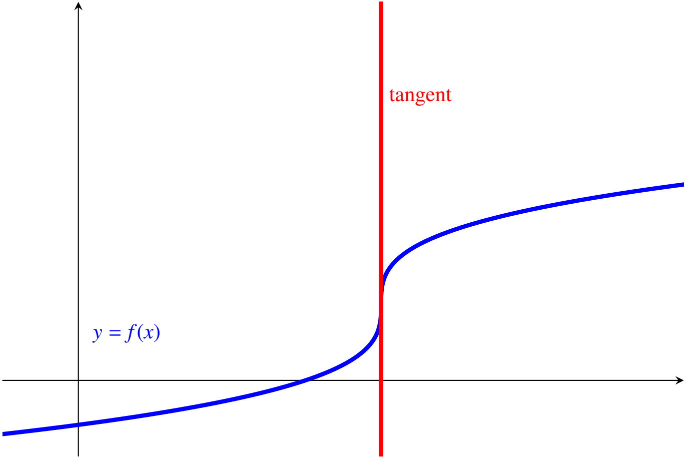

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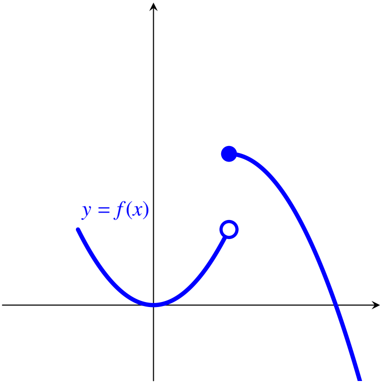
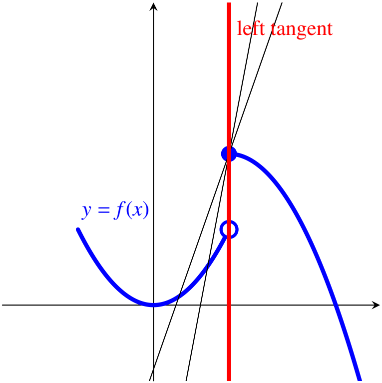


A function is not differentiable at a jump discontinuity, where the one-sided tangents have different (and possibly infinite) slopes

Differentiability



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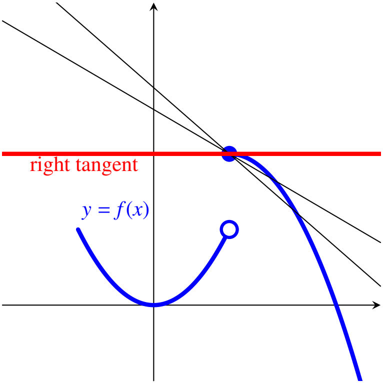

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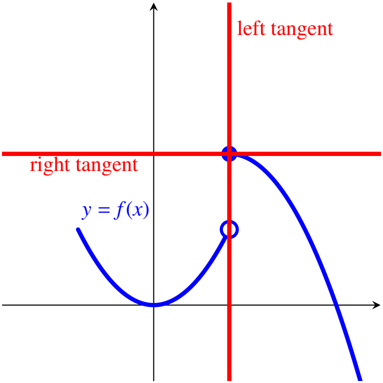
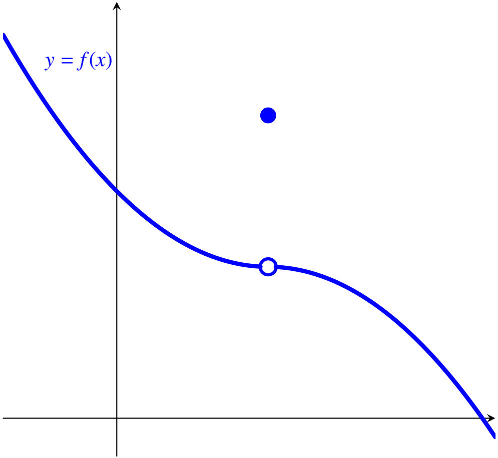
Differentiability



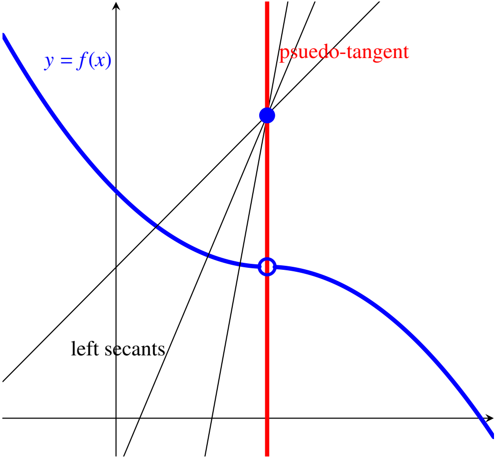
A function is not differentiable at a jump discontinuity, where the one-sided tangents have different (and possibly infinite) slopes

Differentiability



A function is not differentiable at a removable discontinuity, where the secant slopes approach $+\infty$ from one side and $-\infty$ from the other

Differentiability



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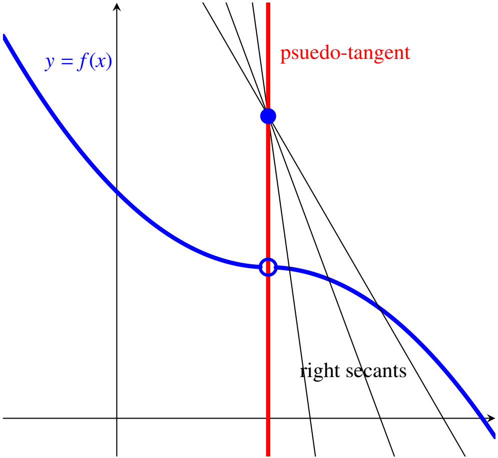
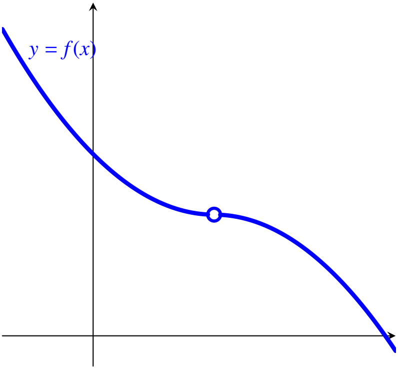


A function is not differentiable at a removable discontinuity, where the secant slopes approach $+\infty$ from one side and $-\infty$ from the other

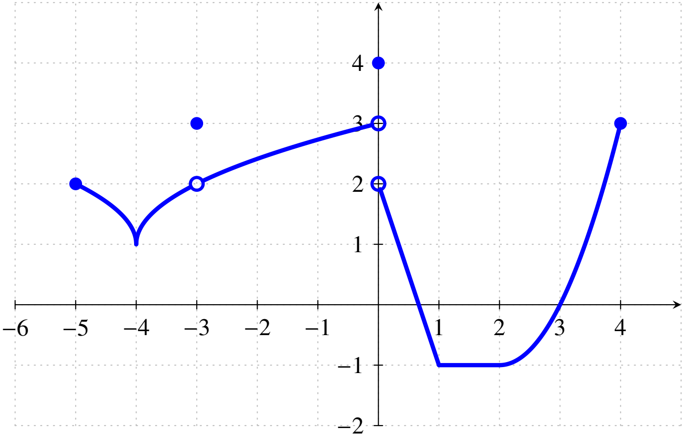
Differentiability



A function is not differentiable at a removable discontinuity where $f(a)$ does not exist since $f(a)$ is required for the tangent slope computation $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Differentiability

- Find the x in $[-5, 4]$ for which $y = f(x)$ is **not** continuous.
- Find the x in $[-5, 4]$ for which $y = f(x)$ is **not** differentiable.
- Estimate $f'(3)$.



Differentiability implies continuity

Theorem

If $f(x)$ has a derivative at $x = a$, then $f(x)$ is continuous at $x = a$.

Note: the converse is **not** true. There are continuous functions that are not differentiable.

Differentiability implies continuity

Claim: If $f'(a)$ exists, then f is continuous at a.

Proof: For $x \neq a$,

$$f(x) = f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limit as $x \to a$,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a} \right).$$

Differentiability implies continuity

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Taking the limit as $x \to a$,

$$\lim_{x \to a} f(x) = f(a) + 0 \cdot f'(a).$$

Differentiability implies continuity

Claim: If $f'(a)$ exists, then f is continuous at a.

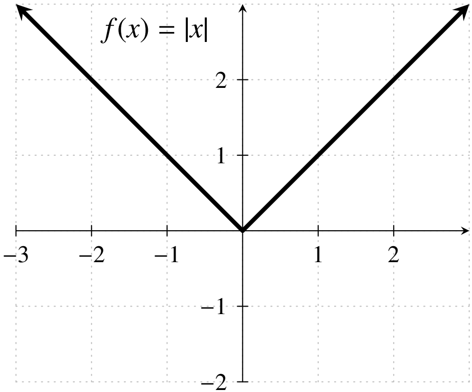
Proof: For $x \neq a$,

$$f(x) = f(a) + (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limit as $x \to a$,

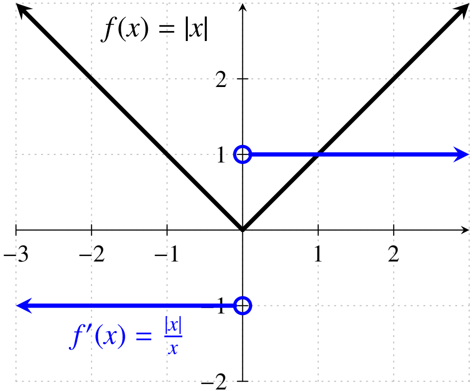
$$\lim_{x \to a} f(x) = f(a).$$

Continuous but not differentiable



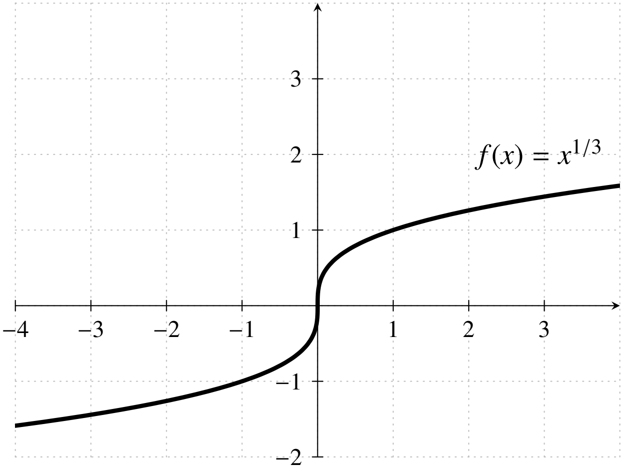
Corner at $x = 0$.

Continuous but not differentiable



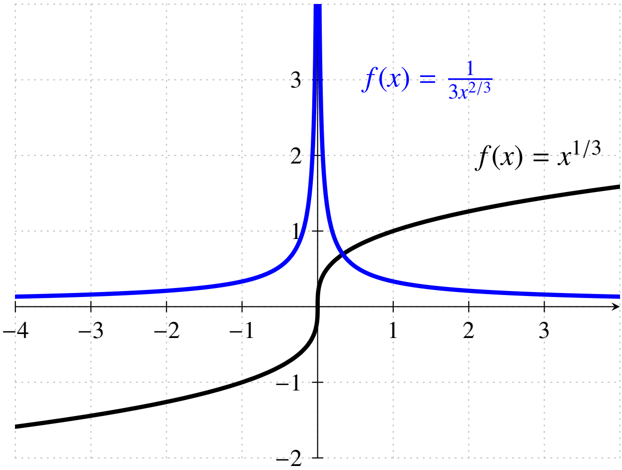
Corner at $x = 0$.

Continuous but not differentiable



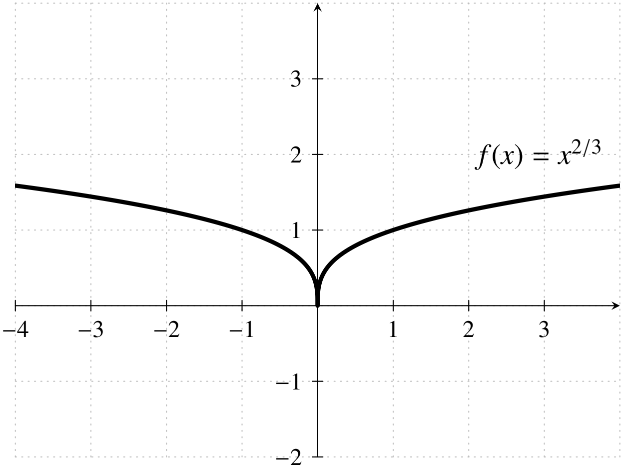
Vertical tangent at $x = 0$.

Continuous but not differentiable



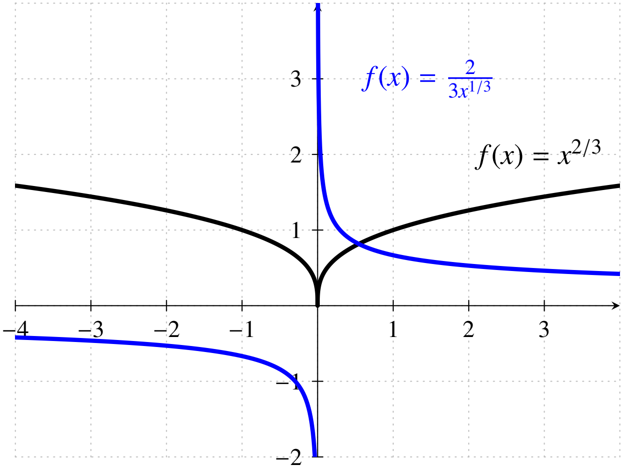
Vertical tangent at $x = 0$.

Continuous but not differentiable



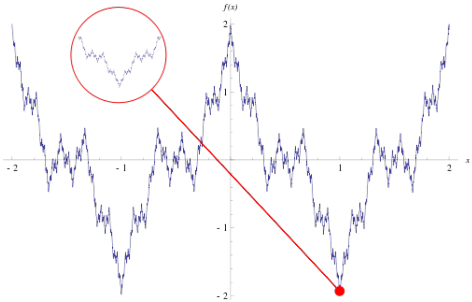
Cusp and vertical tangent at $x = 0$.

Continuous but not differentiable



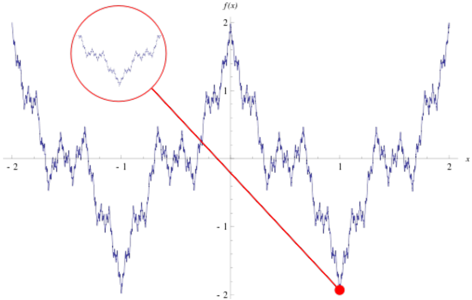
Cusp and vertical tangent at $x = 0$.

Weierstrass Function Karl Weierstrass constructed a function that is continuous everywhere but differentiable nowhere.



Plot of a Weierstrass Function over the interval $[-2, 2]$. Like fractals, the function exhibits self-similarity: every zoom (red circle) is similar to the global plot.

Weierstrass Function Karl Weierstrass constructed a function that is continuous everywhere but differentiable nowhere.



$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$.

Basic trig derivatives

Remark

Recall the angle addition formulas from trigonometry:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

and the limits we encountered in chapter 1:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

These are the key ingredients in the proof of the following theorem.

Theorem

- $\boxed{\dfrac{d}{dx}(\sin x) = \cos x}$
- $\boxed{\dfrac{d}{dx}(\cos x) = -\sin x}$

Example

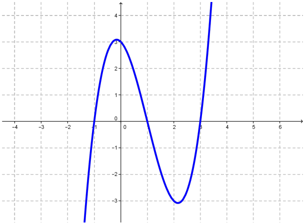
Find the derivative of $f(x) = \begin{cases} \cos x & x \geq 0 \\ 1 & x < 0 \end{cases}$

Remark

1. A continuous function can be described as one whose graph we could sketch without lifting our pencil.
2. A differentiable function can be described as a continuous function that does not have any "sharp corners."

Just checking. . . .

1. Find the derivative of $f(x) = \dfrac{1}{x}$.
2. Find equations of the tangent and normal lines to $y = 1/x$ at $(2, 1/2)$.
3. Approximate the value of the derivative $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ for $f(x) = e^x$ by taking $h = 0.1$.
4. The approximation above is an [overestimate | underestimate] of the true value of $f'(0)$ for $f(x) = e^x$. (Hint: think graphically)
5. Sketch the graph of the derivative of the function shown at right.



A. Instantaneous rate of change

Example

1. Let $P(t)$ represent the world population t minutes after midnight on January 1, 2012. Given that $P(0) = 7,028,734,178$ and that $P'(0) = 156$, estimate the population at the end of the month.
2. Let $M(v)$ represent the mileage (in mpg) of a car traveling at speed v (in mi/h). If $M(55) = 28$ and $M'(55) = -0.2$, estimate $M(65)$.

Remark

These examples utilize the approximation

$$f'(c) \approx \frac{f(c+h) - f(c)}{h} \iff f(c+h) \approx f(c) + f'(c) \cdot h$$

A. Instantaneous rate of change

Remark

Let $s(t)$ represent the position s of an object moving in a straight line at time t.

1. The velocity of the object is $v(t) = s'(t)$.
2. The acceleration of the object is $a(t) = v'(t) = s''(t)$.

Example

Let $s(t) = t^2 - 3t$ describe the position (in m) of an object moving along a straight line as a function of time (in sec).

1. What is its position at $t = 2$? ______________________
2. What is its velocity at $t = 2$? ______________________
3. What is its acceleration at $t = 2$? ____________________
4. When is the object moving forward? __________________

A. Instantaneous rate of change

Remark

Let $s(t)$ represent the position s of an object moving in a straight line at time t.

1. The velocity of the object is $v(t) = s'(t)$.
2. The acceleration of the object is $a(t) = v'(t) = s''(t)$.

Example

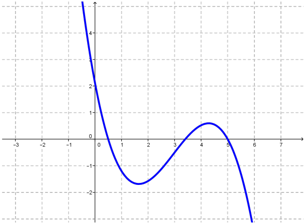
Let $s(t) = t^2 - 3t$ describe the position (in m) of an object moving along a straight line as a function of time (in sec).

1. What is its position at $t = 2$? $s(2) = -2$ m
2. What is its velocity at $t = 2$? $v(2) = 1$ m/s
3. What is its acceleration at $t = 2$? $a(2) = 2$ m/s^2
4. When is the object moving forward? $(1.5, \infty)$

B. Slope of the tangent line

Thinking of the derivative as the slope of a tangent line allows us to:

1. Compare (instantaneous) rates of change
   - e.g. How much faster is $x^2 + 1$ changing at $x = 2$ than at $x = 1$?
2. Sketch a graph of the derivative given a graph of the function
   - e.g. Suppose the graph of the function is



3. Approximate the value of functions
   - e.g. Estimate $\sqrt{5}$

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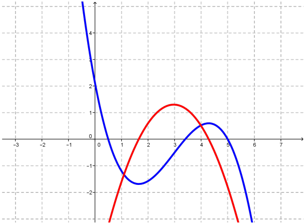


3. Approximate the value of functions
   - e.g. Estimate $\sqrt{5}$

Just checking. . . .

1. What functions have a constant rate of change?
2. Given $f(5) = 9$ and $f'(5) = -0.3$, approximate $f(6)$.
3. The height H (in feet) of Lake Macatawa is is recorded t hours after midnight on May 1. What are the units of $H'(t)$? What does $H'(17) = -1/120$ mean?
4. Numerically approximate the value of $f'(4)$ for $f(x) = \ln x$.
5. Use the definition of the derivative to compute $f'(x)$ for $f(x) = (x - 2)^3$.

Example

Using a graph of the function (whenever possible) or the defintion of the derivative (whenever necesary), find the derivatives of the following functions.

1. $f(x) = x^0$

   $f'(x) =$ ______________________

2. $f(x) = x$

   $f'(x) =$ ______________________

3. $f(x) = x^2$

   $f'(x) =$ ______________________

4. $f(x) = x^3$

   $f'(x) =$ ______________________

5. $f(x) = 1/x$

   $f'(x) =$ ______________________

6. $f(x) = 3x$

   $f'(x) =$ ______________________

7. $f(x) = 3x + 1$

   $f'(x) =$ ______________________

8. $f(x) = e$

   $f'(x) =$ ______________________

Example

Using a graph of the function (whenever possible) or the defintion of the derivative (whenever necesary), find the derivatives of the following functions.

1. $f(x) = x^0$

   $f'(x) = 0$
2. $f(x) = x$

   $f'(x) = 1$
3. $f(x) = x^2$

   $f'(x) = 2x$
4. $f(x) = x^3$

   $f'(x) = 3x^2$
5. $f(x) = 1/x$

   $f'(x) = -1/x^2$
6. $f(x) = 3x$

   $f'(x) = 3$
7. $f(x) = 3x + 1$

   $f'(x) = 3$
8. $f(x) = e$

   $f'(x) = 0$

Theorem

Basic differentiation rules

1. $\dfrac{d}{dx}(c) = 0$
2. $\dfrac{d}{dx}(x^n) = nx^{n-1}$
3. $\dfrac{d}{dx}(\sin x) = \cos x$
4. $\dfrac{d}{dx}(\cos x) = -\sin x$
5. $\dfrac{d}{dx}(e^x) = e^x$
6. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Theorem

Basic differentiation properties

1. $\dfrac{d}{dx}\Big[f(x) \pm g(x)\Big] = f'(x) \pm g'(x)$
2. $\dfrac{d}{dx}\Big[c \cdot f(x)\Big] = c \cdot f'(x)$

Remark

Differentiation respects addition and constant multiples because derivatives are limits (of difference quotients) and limits respect addition and constant multiples:

- $\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x)$

But derivatives do not inherit all the properties of limits (as we'll see in the next section) because those limit laws concerned limits of functions, whereas derivatives are limits of difference quotients. For instance, the limit of a product (or quotient) is the product (or quotient) of the limits, but the derivative of a product (or quotient) is *not* the product (or quotient) of the derivatives (as we'll see in the next section).

Example

Find the derivatives of the following functions.

1. $f(x) = 2x^2 - 3x + 1$
   $f'(x) =$ ____________________
2. $g(x) = 3e^x + 2\sin x$
   $g'(x) =$ ____________________

Find an equation for the line tangent to

1. f at $x = 3$
   ____
2. g at $x = 0$
   ____

Now

1. Without using any calculus approximate $g(0.1) \approx$ ____________
2. Approximate $g(0.1)$ by using an appropriate tangent line.
   $g(0.1) \approx$ ____________________

Example

Find the derivatives of the following functions.

1. $f(x) = 2x^2 - 3x + 1$

   $f'(x) = 4x - 3$

2. $g(x) = 3e^x + 2\sin x$

   $g'(x) = 3e^x + 2\cos x$

Find an equation for the line tangent to

1. f at $x = 3$

   $y - 10 = 9(x - 3)$

2. g at $x = 0$

   $y - 3 = 5x$

Now

1. Without using any calculus approximate $g(0.1) \approx g(0) = 3$
2. Approximate $g(0.1)$ by using an appropriate tangent line.

   $g(0.1) \approx 3.5$ (Cf. $g(0.1) = 3.515$)

Example

Let $f(x) = \cos x + x/2 - 1$.

1. Approximate $f(3)$ without using any calculus.

   $f(3) \approx$ ____________________________________

2. Now approximate $f(3)$ by using an appropriate tangent line.

   $f(3) \approx$ ____________________________________

3. Find the value(s) of x, if any, where f has a horizontal tangent.

   $x =$ ____________________________________

Example

Let $f(x) = \cos x + x/2 - 1$.

1. Approximate $f(3)$ without using any calculus.

   $f(3) \approx f(\pi) = \pi/2 - 2 \approx -0.43$

2. Now approximate $f(3)$ by using an appropriate tangent line.

   $f(3) \approx -0.5$ (Cf. $f(3) = -0.490$)

3. Find the value(s) of x, if any, where f has a horizontal tangent.

   $x = \ldots, -7\pi/6, \pi/6, 5\pi/6, \ldots$ (i.e. whenever $\sin x = 1/2$)

Higher order derivatives

Definition

Let $y = f(x)$ be differentiable on an interval I.

1. The *second derivative* of f is

$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = \frac{d^2y}{dx^2} = y''$$

2. The *third derivative* of f is

$$f'''(x) = \frac{d}{dx}\Big(f''(x)\Big) = \frac{d}{dx}\Big(\frac{d^2y}{dx^2}\Big) = \frac{d^3y}{dx^3} = y'''$$

3. The n^{th} *derivative* of f is

$$f^{(n)}(x) = \frac{d}{dx}\Big(f^{(n-1)}(x)\Big) = \frac{d}{dx}\Big(\frac{d^{(n-1)}y}{dx}\Big) = \frac{d^ny}{dx^n} = y^{(n)}$$

Higher order derivatives

Example

Find the first four derivatives of the following functions.

1. $f(x) = 3x^2 - 2x$

   $f'(x) = \_\_\_\_\_\_\_\_$

   $f''(x) = \_\_\_\_\_\_\_\_$

   $f'''(x) = \_\_\_\_\_\_\_\_$

   $f^{(4)}(x) = \_\_\_\_\_\_\_\_$

2. $f(x) = 3e^x$

   $f'(x) = \_\_\_\_\_\_\_\_$

   $f''(x) = \_\_\_\_\_\_\_\_$

   $f'''(x) = \_\_\_\_\_\_\_\_$

   $f^{(4)}(x) = \_\_\_\_\_\_\_\_$

3. $f(x) = \cos x$

   $f'(x) = \_\_\_\_\_\_\_\_$

   $f''(x) = \_\_\_\_\_\_\_\_$

   $f'''(x) = \_\_\_\_\_\_\_\_$

   $f^{(4)}(x) = \_\_\_\_\_\_\_\_$

Higher order derivatives

Example

Find the first four derivatives of the following functions.

1. $f(x) = 3x^2 - 2x$

   $f'(x) = 6x - 2$

   $f''(x) = 6$

   $f'''(x) = 0$

   $f^{(4)}(x) = 0$

2. $f(x) = 3e^x$

   $f'(x) = 3e^x$

   $f''(x) = 3e^x$

   $f'''(x) = 3e^x$

   $f^{(4)}(x) = 3e^x$

3. $f(x) = \cos x$

   $f'(x) = -\sin x$

   $f''(x) = -\cos x$

   $f'''(x) = \sin x$

   $f^{(4)}(x) = \cos x$

Higher order derivatives

Remark

The second derivative is the rate of change of the rate of change of the function; or, put geometrically, the rate of change of the slope of tangent lines.

- The sign of the first derivative tells us whether the function is changing positively (i.e. increasing) or negatively (i.e. decreasing). So, the sign of the first derivative tells us *whether* the function is increasing or decreasing.
- The sign of the second derivative tells us whether the tangent slopes are changing positively (i.e increasing) or negatively (i.e. decreasing). So, the sign of the first derivative tells us *how* the function is increasing or decreasing.

Just checking. . . .

1. Differentiate whichever functions you can using the differentiation rules we've discussed so far.

   a. $f(x) = 3/x^2$
   b. $g(x) = 3/(x+1)^2$
   c. $h(x) = (3x^2+1)/x^3$
   d. $i(x) = 3x^3/(x^3+2)$
   e. $j(x) = \sqrt[3]{x}$
   f. $k(x) = \sqrt{x+1}$
   g. $\ell(x) = \sqrt[4]{x} + 3$
   h. $m(x) = 2e^x$
   i. $n(x) = e^{2x}$
   j. $p(x) = xe^2$
   k. $q(x) = \ln(x^2)$
   l. $r(x) = 2\sin x$
   m. $s(x) = \sin(2x)$
   n. $t(x) = \sin x \cos x$

2. Where does the line that is tangent to $f(x) = e^x + 3$ at $x = 0$ intersect the x-axis?

3. Approximate $e^{0.1}$ using an appropriate tangent line.

Theorem

(The Product Rule)
Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Example

Find the derivatives of the following functions.

1. $f(x) = 3x \sin x$

   $f'(x) =$ ____________________

2. $f(x) = xe^x$

   $f'(x) =$ ____________________

3. $f(x) = x \ln x - x$

   $f'(x) =$ ____________________

4. $f(x) = (x + 1)(3x^2 - 2)$

   $f'(x) =$ ____________________

Theorem

(The Product Rule)
Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Example

Find the derivatives of the following functions.

1. $f(x) = 3x\sin x$

   $f'(x) = 3\sin x + 3x\cos x$
2. $f(x) = xe^x$

   $f'(x) = e^x + xe^x$
3. $f(x) = x\ln x - x$

   $f'(x) = \ln x$
4. $f(x) = (x+1)(3x^2-2)$

   $f'(x) = 9x^2 + 6x - 2$

Theorem

(The Quotient Rule)

Let f and g be differentiable functions on an open interval I, and suppose $g(x) \neq 0$ for all x in I. Then f/g is differentiable on I and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example

Find the derivatives of the following functions.

1. $f(x) = (x^2 + 3)/x$
2. $f'(x) =$ ____________________
3. $f(x) = (x^2 + 3)/(x + 1)$
4. $f'(x) =$ ____________________
5. $f(x) = x^2/(x + 1)$
6. $f'(x) =$ ____________________
7. $f(x) = \tan x$
8. $f'(x) =$ ____________________

Theorem

(The Quotient Rule)
Let f and g be differentiable functions on an open interval I, and suppose $g(x) \neq 0$ for all x in I. Then f/g is differentiable on I and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example

Find the derivatives of the following functions.

1. $f(x) = (x^2 + 3)/x$
2. $f'(x) = 1 - 3/x^2$
3. $f(x) = (x^2 + 3)/(x + 1)$
4. $f'(x) = \frac{x^2+2x-3}{(x+1)^2}$
5. $f(x) = x^2/(x + 1)$
6. $f'(x) = \frac{x^2+2x}{(x+1)^2}$
7. $f(x) = \tan x$
8. $f'(x) = \sec^2 x$

Theorem

Derivatives of trigonometric functions

1. $\dfrac{d}{dx}\left(\sin x\right) = \cos x$
2. $\dfrac{d}{dx}\left(\tan x\right) = \sec^2 x$
3. $\dfrac{d}{dx}\left(\sec x\right) = \sec x \tan x$
4. $\dfrac{d}{dx}\left(\cos x\right) = -\sin x$
5. $\dfrac{d}{dx}\left(\cot x\right) = -\csc^2 x$
6. $\dfrac{d}{dx}\left(\csc x\right) = -\csc x \cot x$

Just checking. . . .

1. True or false. The derivatives of the trigonometric "co-" functions (i.e. the ones that start with "co-") have minus signs in them.
2. Find the values of x in $[-1, 1]$ where the tangent line to $f(x) = x \sin x$ is horizontal.
3. Find an equation of the normal line to $f(x) = \frac{x^2}{x-1}$ at $(2, 4)$.
4. Find the derivative of $xe^x \sin x$.
5. Find the derivative of $\sin x \csc x$.

Example

Find the derivatives of the following functions.

1. $F_2(x) = (1 - x)^2$. $F_2'(x) =$ ______________________
2. $F_3(x) = (1 - x)^3$. $F_3'(x) =$ ______________________
3. $F_4(x) = (1 - x)^4$. $F_4'(x) =$ ______________________

Remark

Notice that each of the functions is a composition $F_n(x) = f_n(g(x))$, where the "inner piece" is $g(x) = 1 - x$ and the "outer piece" is $f_n(x) = x^n$ (for $n = 2, 3, 4$). Notice, too, that we can differentiate each "piece." The chain rule tells us how to put the derivatives of these pieces together to get the derivative of a composition.

Example

Find the derivatives of the following functions.

1. $F_2(x) = (1-x)^2$. $F_2'(x) = -2(1-x)$
2. $F_3(x) = (1-x)^3$. $F_3'(x) = -3(1-x)^2$
3. $F_4(x) = (1-x)^4$. $F_4'(x) = -4(1-x)^3$

Remark

Notice that each of the functions is a composition $F_n(x) = f_n(g(x))$, where the “inner piece” is $g(x) = 1 - x$ and the “outer piece” is $f_n(x) = x^n$ (for $n = 2, 3, 4$). Notice, too, that we can differentiate each “piece.” The chain rule tells us how to put the derivatives of these pieces together to get the derivative of a composition.

Theorem

(The Chain Rule)

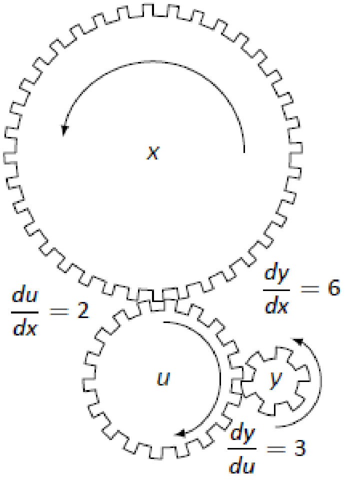
Let $y = f(u)$ be a differentiable function of u, and let $u = g(x)$ be a differentiable function of x.

Then $y = f(g(x))$ is a differentiable function of x and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

or in Newton's notation

$$\boxed{\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x)}$$



Example

Find the derivatives of the following functions.

1. $y = \cos(3x)$

   $dy/dx =$ ____________________

2. $y = \sin^{17} x$

   $dy/dx =$ ____________________

3. $y = \ln(x^{-2})$

   $dy/dx =$ ____________________

4. $y = e^{x^2}$

   $dy/dx =$ ____________________

Theorem

Let $u = u(x)$ be a differentiable function of x. Then:

1. $\frac{d}{dx}(u^n) = nu^{n-1} \cdot (du/dx)$
2. $\frac{d}{dx}(e^u) = e^u \cdot (du/dx)$
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot (du/dx)$
4. $\frac{d}{dx}(\sin u) = \cos u \cdot (du/dx)$
5. $\frac{d}{dx}(\cos u) = -\sin u \cdot (du/dx)$
6. $\frac{d}{dx}(\tan u) = \sec^2 u \cdot (du/dx)$

Example

Find the derivatives of the following functions.

1. $y = \cos(3x)$

   $dy/dx = -3\sin(3x)$
2. $y = \sin^{17} x$

   $dy/dx = 17\sin^{16} x \cos x$
3. $y = \ln(x^{-2})$

   $dy/dx = -2x$
4. $y = e^{x^2}$

   $dy/dx = 2xe^{x^2}$

Theorem

Let $u = u(x)$ be a differentiable function of x. Then:

1. $\frac{d}{dx}(u^n) = nu^{n-1} \cdot (du/dx)$
2. $\frac{d}{dx}(e^u) = e^u \cdot (du/dx)$
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot (du/dx)$
4. $\frac{d}{dx}(\sin u) = \cos u \cdot (du/dx)$
5. $\frac{d}{dx}(\cos u) = -\sin u \cdot (du/dx)$
6. $\frac{d}{dx}(\tan u) = \sec^2 u \cdot (du/dx)$

Example

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin(3x)$

   $f'(x) =$ ______________________

2. $f(x) = \dfrac{3x+1}{\sin^3 x}$

   $f'(x) =$ ______________________

3. $f(x) = \cos(\sqrt{1-2x})$

   $f'(x) =$ ______________________

4. $f(x) = \ln(2e^{\sin x})$

   $f'(x) =$ ______________________

Remark

Recall that $e^{\ln u} = u$. So $a^x = e^{\ln a^x} = e^{x \ln a}$.

Example

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin(3x)$
   $f'(x) =$
   $5x^4 \sin(3x) + 3x^5 \cos(3x)$
2. $f(x) = \dfrac{3x+1}{\sin^3 x}$
   $f'(x) =$
   $\dfrac{3\sin^3 x - 3x\sin^3 x - \sin^3 x}{\sin^6 x}$
3. $f(x) = \cos(\sqrt{1-2x})$
   $f'(x) = \dfrac{\sin\sqrt{1-2x}}{\sqrt{1-2x}}$
4. $f(x) = \ln(2e^{\sin x})$
   $f'(x) = \cos x$

Remark

Recall that $e^{\ln u} = u$. So $a^x = e^{\ln a^x} = e^{x \ln a}$.

Theorem

Let $a > 0$ (and $a \neq 1$). Then

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

More generally, if $u = u(x)$ is a differentiable function of x, then

$$\frac{d}{dx}(a^u) = a^u(\ln u) \cdot \frac{du}{dx}$$

Also,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

and more generally

$$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \cdot \frac{du}{dx}$$

Just checking. . . .

1. Compute $\frac{d}{dx}(\ln(kx))$ in two ways: (a) by using the chain rule, and (b) by using laws of logs first, and then differentiating.
2. Compute $\frac{d}{dx}\left(\ln(x^k)\right)$ in two ways: (a) by using the chain rule, and (b) by using laws of logs first, and then differentiating.
3. True or false. $\frac{d}{dx}(3^x) \approx (1.1)3^x$.
4. Compute $\frac{d}{dx}\left(\cos(1/x)e^{5x^2}\right)$.
5. Find equations for the tangent and normal lines to $g(x) = (\sin x + \cos x)^3$ at $x = \pi/2$.

Example

1. If $y = x$, what is $\frac{d}{dx}\left(y^3\right)$?
2. If $y = \cos x$, what is $\frac{d}{dx}\left(y^3\right)$?
3. If $y = y(x)$, what is $\frac{d}{dx}\left(y^3\right)$?

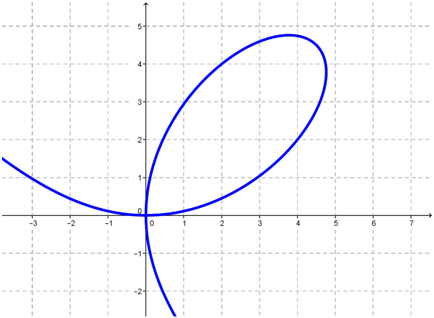
Remark

- When we know how y is related to x, we can differentiate any function of y with respect to x using the chain rule.
- When we do *not* know how y is related to x, we can still differentiate any function of y with respect to x using the chain rule – but the exact dependence of y on x (i.e. dy/dx) will remain implicit, since we can't determine explicitly what dy/dx is.

Example

Any equation in x and y determines a curve in the xy-plane: the graph of an equation in x and y is just the set of points (x, y) that satisfy the equation. For example, the graph of the equation $x^3 + y^3 = 9xy$ is shown below at right.

1. Verify that $(2, 4)$ lies on the curve $x^3 + y^3 = 9xy$.
2. Use the graph to estimate $\left.\frac{dy}{dx}\right|_{(2,4)}$.
3. Treating y as an unknown function of x locally, which we write as $[y = y(x)]$, compute $\left.\frac{dy}{dx}\right|_{(2,4)}$.



$x^3 + y^3 = 9xy$

Example

Find dy/dx for the following curves.

1. $x^{2/5} + y^{2/5} = 1$

   $dy/dx =$ ____________________

2. $\dfrac{x^2 + y}{x + y^2} = 17$

   $dy/dx =$ ____________________

3. $\ln(x^2 + xy + y^2) = 1$

   $dy/dx =$ ____________________

4. $\dfrac{\cos(x^2y^2) + y^3}{x + y} = 1$

   $dy/dx =$ ____________________

Example

Find an equation for the line tangent to the circle $(x - 1)^2 + (y + 2)^2 = 1$ at the point $(3/2, \sqrt{3}/2 - 2)$ in two ways:

- by using implicit differentiation
- by first solving for y as a function of x and then differentiating

Example

Find dy/dx for the following curves.

1. $x^{2/5} + y^{2/5} = 1$

   $dy/dx = -\dfrac{y^{3/5}}{x^{3/5}}$

2. $\dfrac{x^2 + y}{x + y^2} = 17$

   $dy/dx = \dfrac{2x - 17}{34y - 1}$

3. $\ln(x^2 + xy + y^2) = 1$

   $dy/dx = -(2x + y)/(x + 2y)$

4. $\dfrac{\cos(x^2y^2) + y^3}{x + y} = 1$

   $dy/dx =$
   $-\dfrac{2xy^2\sin(x^2y^2) + 1}{2x^2y\sin(x^2y^2) - 3y + 1}$

Example

Find an equation for the line tangent to the circle $(x - 1)^2 + (y + 2)^2 = 1$ at the point $(3/2, \sqrt{3}/2 - 2)$ in two ways:

- by using implicit differentiation
- by first solving for y as a function of x and then differentiating

Theorem

Let r be a real number. Then

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}$$

Implicit differentiation and second derivatives

Example

Find d^2y/dx^2 where $x^2 + y^2 = 4$.

Logarithmic differentiation

Example

Find dy/dx for the following functions.

1. $y = \dfrac{(x-2)^3\sqrt{3x+1}}{(2x+5)^4}$
2. $y = x^{\sin x}$

Just checking. . . .

1. Find dy/dx for the following implicitly defined functions.

   a. $xy = 1$
   b. $x^2y^2 = 1$
   c. $\sin(xy) = 1$
   d. $\ln(xy) = 1$

2. Find dy/dx for $x^2 \tan y = 50$.
3. Find an equation for the line tangent to $y = (2x)^{x^2}$ at $x = 1$.
4. Find d^2y/dx^2 for $\cos x + \sin y = 1$.
5. Using the defintion of the derivative, compute $f'(x)$ for $f(x) = \sqrt{3x+1}$, and write an equation for the line tangent to $y = f(x)$ at $(1, 2)$.

Background

Definition

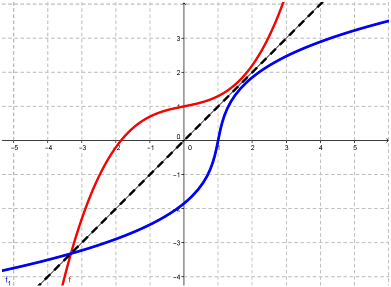
Two functions f and g are inverses of each other provided

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

We write $f^{-1}(x)$ for the inverse of f.

Remark

If g is the inverse of f, then g "undoes" whatever f does: if $f(a) = b$, then $g(b) = a$. A consequence of this is that the graph of the inverse function $y = f^{-1}(x)$ is the reflection of the graph of the function $y = f(x)$ through the line $y = x$.



Background

Remark

A function takes each input to a single output: $f(a) = b$. Since the inverse function will take b back to a, we need f to take *only one* input to b; otherwise, an inverse function cannot be defined.

Definition

A function f is *injective* (or *one-to-one*) if distinct inputs get sent to distinct outputs:

- If $a \neq b$, then $f(a) \neq f(b)$

  – or equivalently –

- If $f(a) = f(b)$, then $a = b$.

Background

Theorem

A function f has an inverse if and only if f is injective.

Remark

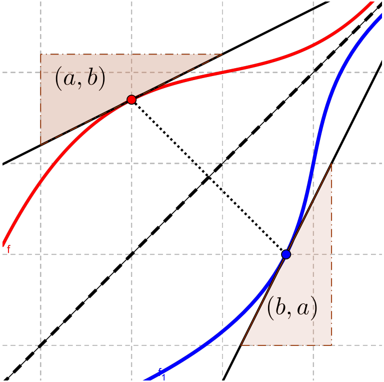
If f is injective, then a horizontal line will intersect the graph of f at most once, and vice versa.

Example

Given that $y = \dfrac{2x - 3}{x + 1}$ is injective, find the inverse function. (Can you show that this function is injective?)

Remark

Recall that a function and its inverse are reflections of each other through the line $y = x$. A consequence of this is that the derivative of the inverse at a point is the reciprocal of the derivative of the function at the corresponding point.



Theorem

Let f be a differentiable and injective function, and let $g = f^{-1}$ be the inverse of f. Suppose $f(a) = b$ so that $g(b) = a$. Then

$$\left(f^{-1}\right)'(b) = g'(b) = \frac{1}{f'(a)}$$

and more generally

$$\left(f^{-1}\right)'(x) = g'(x) = \frac{1}{f'(g(x))}$$

Example

$g(x) = \arcsin x$ and $f(x) = \sin x$ are inverses, and so the theorem above gives

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)} \quad (???)$$

Inverse trig functions

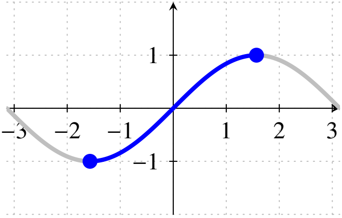
Inverse trig functions (sans domain restriction) are defined by:

$$\begin{aligned} y = \arcsin(x) \quad &\Leftrightarrow \quad \sin(y) = x. \\ y = \arccos(x) \quad &\Leftrightarrow \quad \cos(y) = x. \\ y = \arctan(x) \quad &\Leftrightarrow \quad \tan(y) = x. \end{aligned}$$

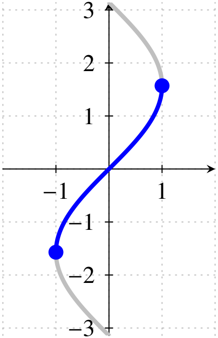
Caution: Inverse trig functions are not reciprocal functions.

$$\begin{aligned} \arcsin(x) = \operatorname{asin}(x) = \sin^{-1}(x) \quad &\neq \quad \tfrac{1}{\sin(x)} = \csc(x). \\ \arccos(x) = \operatorname{acos}(x) = \cos^{-1}(x) \quad &\neq \quad \tfrac{1}{\cos(x)} = \sec(x). \\ \arctan(x) = \operatorname{atan}(x) = \tan^{-1}(x) \quad &\neq \quad \tfrac{1}{\tan(x)} = \cot(x). \end{aligned}$$

Arcsine

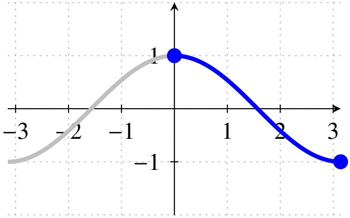


$y = \sin(x)$
$-\pi/2 \le x \le \pi/2$
$-1 \le y \le 1$

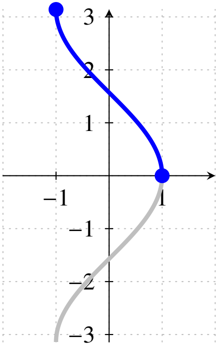


$y = \arcsin(x)$
$-1 \le x \le 1$
$-\pi/2 \le y \le \pi/2$

Arccosine

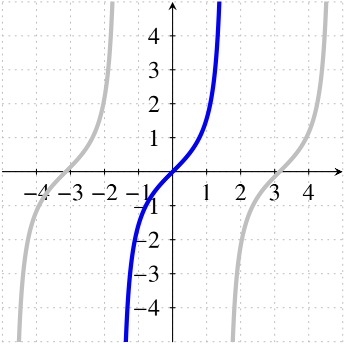


$y = \cos(x)$
$0 \leq x \leq \pi$
$-1 \leq y \leq 1$



$y = \arccos(x)$
$-1 \leq x \leq 1$
$0 \leq y \leq \pi$

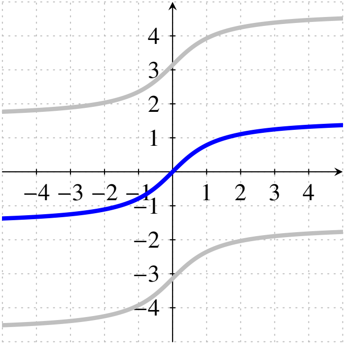
Arctangent



$y = \tan(x)$

$-\pi/2 < x < \pi/2$

$-\infty < y < \infty$



$y = \arctan(x)$

$-\infty < x < \infty$

$-\pi/2 < y < \pi/2$

Derivatives of inverse trig functions

Example

Find $\frac{d}{dx}\left(\arccos(3x)\right)$.

Example

Find $\frac{d}{dx}\left(\arcsin(2x^3)\right) = $ ________________________________

Remark

This technique can be used to show $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Derivatives of inverse trig functions

Example

Find $\frac{d}{dx}(\arccos(3x))$.

Example

Find $\frac{d}{dx}\left(\arcsin(2x^3)\right) = \dfrac{6x^2}{\sqrt{1-4x^6}}$

Remark

This technique can be used to show $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Just checking. . . .

1. If $(3, 4)$ lies on the graph of $y = f(x)$ and $f'(3) = -5$, what can be said about the graph of $y = f^{-1}(x)$?
2. Find an equation of the line that is tangent to $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.
3. Compute the derivative of the function $f(x) = \sin(\cos^{-1} x)$ in two ways:
   a. by using the chain rule first, and then simplifying
   b. by simplifying first, and then taking the derivative
4. Find an equation of the line that is tangent to the function $f(x) = \sin^{-1}(2x)$ at $x = 1/4$.
5. Find the derivative of $y = \dfrac{(x+3)^7(x-2)^3}{\sqrt[3]{2x-5}}$.