Definition

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum** (or **absolute minimum**) of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum** (or **absolute maximum**) of f on I if $f(c) \geq f(x)$ for all x in I.

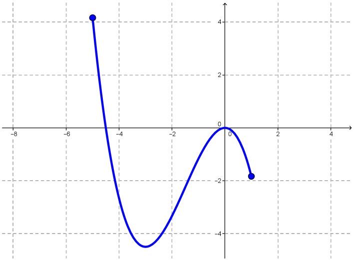
The maximum and minimum values are the **extreme values** (or **extrema**) of f on I. We also call absolute extrema **global extrema**.

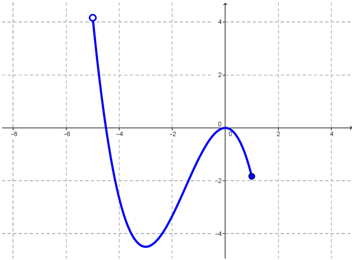
Theorem

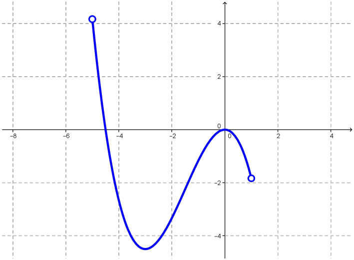
Extreme Value Theorem

Let f be a continuous function defined on a closed interval I. Then f has both a maximum and a minimum value on I.

Example







Definition

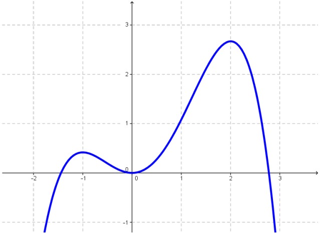
Let f be defined on an interval I containing c.

1. If there is an open interval containing c such that $f(c)$ is the minimum value on that interval, then $f(c)$ is a **relative minimum** or **local minimum** of f.
2. If there is an open interval containing c such that $f(c)$ is the maximum value on that interval, then $f(c)$ is a **relative maximum** or **local maximum** of f.

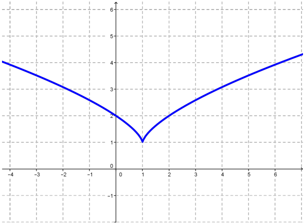
Collectively, relative maxima and minima are called **relative extrema** or **local extrema**.

Example

1. $f(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2$



2. $f(x) = (x-1)^{2/3} + 1$



Definition

Let f be defined at c. The x-value c is a **critical number** or **critical value** of f if either

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

If c is a critical value of f, the point $(c, f(c))$ is a **critical point** of f.

Theorem

If f has a relative extreme value at c, then c is a critical number of f.

Remark

That is, all relative extrema occur at critical values. However, not all critical values give relative extrema.

Remark

Finding extrema on closed intervals

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$ (whose existence is guaranteed by the Extreme Value Theorem), we:

1. Find the critical numbers of f in $[a, b]$.
2. Evaluate f at the critical numbers and at the endpoints.
3. Compare values to find the greatest and least.

Example

Find the extreme values of the following functions on the given intervals, or, if no interval is given, on its domain.

1. $f(x) = x^2 + x + 4$ on $[-1, 2]$

   max: ______________________

   min: ______________________

2. $f(x) = (\ln x)/x$ on $[1, 4]$

   max: ______________________

   min: ______________________

3. $f(x) = e^x \sin x$ on $[0, \pi]$

   max: ______________________

   min: ______________________

4. $f(x) = x^2 \sqrt{4 - x^2}$

   max: ______________________

   min: ______________________

Example

Find the extreme values of the following functions on the given intervals, or, if no interval is given, on its domain.

1. $f(x) = x^2 + x + 4$ on $[-1, 2]$

   max: $f(2) = 10$

   min: $f(-1/2) = 15/4$

2. $f(x) = (\ln x)/x$ on $[1, 4]$

   max: $f(e) = 1/e$

   min: $f(1) = 0$

3. $f(x) = e^x \sin x$ on $[0, \pi]$

   max: $f(3\pi/4) = \sqrt{2}e^{3\pi/4}/2$

   min: $f(0) = f(\pi) = 0$

4. $f(x) = x^2\sqrt{4 - x^2}$

   max: $f(\pm\sqrt{8/3}) = 16/(3\sqrt{3})$

   min: $f(\pm 2) = f(0) = 0$

Just checking. . . .

1. True or false. If c is a critical value of a function f, then f has either a relative maximum or a relative minimum at $x = c$.
2. Find the extreme values of $f(x) = x^3 - \frac{9}{2}x^2 - 30x + 3$ on $[0, 6]$.
3. Let $f(x) = x^3 + x$. Evaluate $\lim_{s \to 0} \dfrac{f(x+s) - f(x)}{s}$.
4. Find $\frac{d}{dx}\left(\sin^{-1}(e^{2x})\right)$.
5. Find $\frac{d}{dx}\left(3^{x^2}\right)$.

Remark

- The average rate of change of f over the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

- The instantaneous rate of change in f at c is $f'(c)$.

Theorem

The Mean Value Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a value of c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of the mean value theorem

Theorem

Rolle's theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b), and suppose $f(a) = f(b) = 0$. Then there is some c in (a, b) such that $f'(c) = 0$.

Proof.

Proof of the mean value theorem

Theorem

The Mean Value Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a value of c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof.

Example

Consider $f(x) = \ln x$ on $[1, 5]$. Find the value of c in $(1, 5)$ where $f'(c)$ equals the average rate of change of $\ln x$ over $[1, 5]$.

Example

Consider $f(x) = (x - 1)^{2/3}$ on $[0, 9]$. Show that there is no value of c in $(0, 9)$ where the instantaneous rate of change in f at c equals the average rate of change in f over $[0, 9]$. Does this contradict the mean value theorem? Why or why not?

Just checking. . . .

1. Find the value of c in $(-5, 2)$ where the average rate of change in $f(x) = 2x^3 - 5x^2 + 6x + 1$ equals the instantaneous rate of change in f at c.
2. Find the extreme values of $f(x) = x^2 - 3x + 9$ on $[-2, 5]$.
3. Find $\frac{d}{dx}\left((\cos x)^{2x}\right)$.
4. Does $f(x) = \frac{x+2}{2x-3}$ have an inverse? If so, find it. If not, state why not.
5. Find $\frac{d}{dx}\left(\sin(e^{\sqrt{3x+5}})\right)$.

Definition

Let f be a function on an interval I.

1. f is **increasing** on I if $f(a) \leq f(b)$ for every $a < b$ in I.
2. f is **decreasing** on I if $f(a) \geq f(b)$ for every $a < b$ in I.

We say that f is *strictly increasing* (or *strictly decreasing*) on I if the inequalities are strict.

Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for every c in (a, b), then f is increasing on $[a, b]$.
2. If $f'(c) < 0$ for every c in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(c) = 0$ for every c in (a, b), then f is constant on $[a, b]$.

Remark

Finding intervals on which f is increasing or decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing or decreasing:

1. Find the critical values of f; that is, the c in I where either $f'(c) = 0$ or $f'(c)$ is undefined.
2. Use the critical values to divide I into subintervals.
3. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   a. If $f'(p) > 0$, then f is increasing on that subinterval.
   b. If $f'(p) < 0$, then f is decreasing on that subinterval.

Example

Find the intervals on which $f(x) = \frac{x^2+15}{x-1}$ is increasing and the intervals on which it is decreasing.

- f is increasing on ______________________________
- f is decreasing on ______________________________

Theorem

First Derivative Test

Let f be differentiable on I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If the sign of f' does not change of c, then $f(c)$ is not a relative minimum or relative maximum of f.

Example

Find the intervals on which $f(x) = \frac{x^2+15}{x-1}$ is increasing and the intervals on which it is decreasing.

- f is increasing on $(-\infty, -3) \cup (5, \infty)$
- f is decreasing on $(-3, 5)$

Theorem

First Derivative Test

Let f be differentiable on I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If the sign of f' does not change of c, then $f(c)$ is not a relative minimum or relative maximum of f.

Example

1. Find and classify the critical points of $f(x) = \frac{x^2+15}{x-1}$.

   Local minima ______

   Local maxima ______

   Neither ______

2. Find and classify the critical points of $f(x) = \frac{(x-2)^{2/3}}{x}$.

   Local minima ______

   Local maxima ______

   Neither ______

3. Find and classify the critical points of $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.

   Local minima ______

   Local maxima ______

   Neither ______

Example

1. Find and classify the critical points of $f(x) = \frac{x^2+15}{x-1}$.

   Local minima $f(5) = 10$

   Local maxima $f(-3) = -6$

   Neither $x = 1$

2. Find and classify the critical points of $f(x) = \frac{(x-2)^{2/3}}{x}$.

   Local minima $f(2) = 0$

   Local maxima $f(6) = 4^{2/3}/6$

   Neither $x = 0$

3. Find and classify the critical points of $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.

   Local minima $f(-3\pi/4) = f(\pi/4) = 1/2$

   Local maxima $f(-\pi/4) = f(3\pi/4) = -1/2$

   Neither *none*

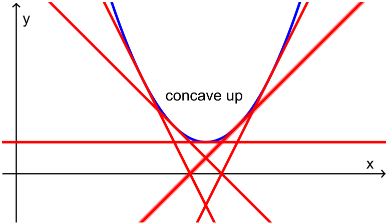
Just checking. . . .

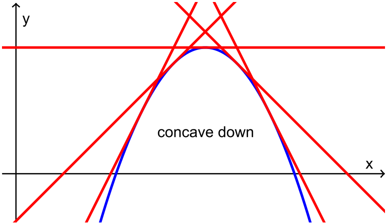
1. Find and classify the critical points of $f(x) = 2x^3 + x^2 - x + 3$.
2. Find the value of c in $(-1, 2)$ where the instantaneous rate of change in $f(x) = x^2 - 3x + 5$ at c is equal to the average rate of chenge in f over $[-1, 2]$.
3. Evaluate $\lim_{x \to 5} \dfrac{x/5 - 5/x}{x - 5}$.
4. Using an $\varepsilon - \delta$ argument, show that $\lim_{x \to 2} 3x^2 = 12$.
5. Find the points on $x^2 - xy + y^2 = 1$ where the tangent line is horizontal and where it is vertical.

Definition

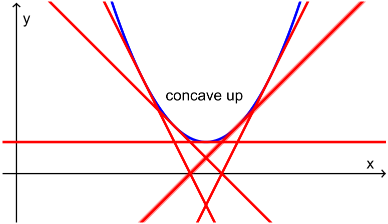
The graph of a function f on an interval

- is **concave up** if it lies above its tangent lines
- is **concave down** if it lies below its tangent lines
- has no concavity if it is flat (i.e., linear)

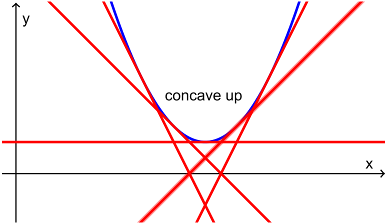




What is happening with the slopes of the tangents when f is concave up? What does this mean for the derivatives of f?



What is happening with the slopes of the tangents when f is concave up? What does this mean for the derivatives of f?

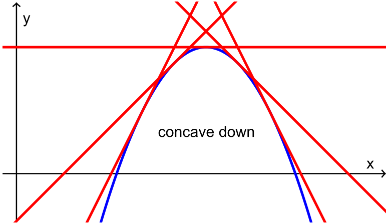


When f is concave up on an interval:

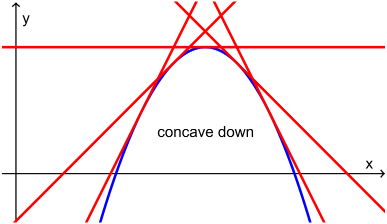
- the slopes of the tangents are increasing.
- $f'(x)$ is increasing.
- $f''(x) > 0$.

Reality check: try $f(x) = x^2$.

What is happening with the slopes of the tangents when f is concave down? What does this mean for the derivatives of f?



What is happening with the slopes of the tangents when f is concave down? What does this mean for the derivatives of f?



When f is concave down on an interval:

- the slopes of the tangents are decreasing.
- $f'(x)$ is decreasing.
- $f''(x) < 0$.

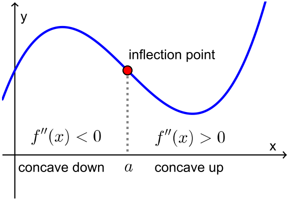
Reality check: try $f(x) = -x^2$.

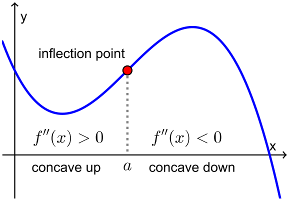
Theorem

Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I and concave down if $f'' < 0$ on I.

Definition

A **point of inflection** is a point on the graph of f at which the concavity of f changes.





Theorem

If $(c, f(c))$ is a point of inflection, then either $f''(c) = 0$ or $f''(c)$ is not defined.

Example

Find the x-coordinates of inflection points (IPs) of the following functions.

1. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x$

   IPs ______________________

2. $f(x) = \frac{x}{x^2-1}$

   IPs ______________________

3. $f(x) = x^2 e^x$

   IPs ______________________

4. $f(x) = x^2 \ln x$

   IPs ______________________

Theorem

Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

Example

Find the x-coordinates of inflection points (IPs) of the following functions.

1. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x$
   IPs $x = 1/3, 1$
2. $f(x) = \frac{x}{x^2-1}$
   IPs $x = -1, 0, 1$
3. $f(x) = x^2 e^x$
   IPs $x = -2 \pm \sqrt{2}$
4. $f(x) = x^2 \ln x$
   IPs $x = e^{-3/2}$

Theorem

Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

Just checking. . . .

1. Is it possible for a function to be
   - increasing and concave down on $(0, \infty)$ with a horizontal asymptote of $y = 1$?
   - increasing and concave up on $(0, \infty)$ with a horizontal asymptote of $y = 1$?
2. Find and classify the critical points of $f(x) = x^2 e^x$.
3. Find the inflection points of $f(x) = x^2 e^x$.
4. Find and classify the critical points of $f(x) = x^2 \ln x$.
5. Find the inflection points of $f(x) = x^2 \ln x$.

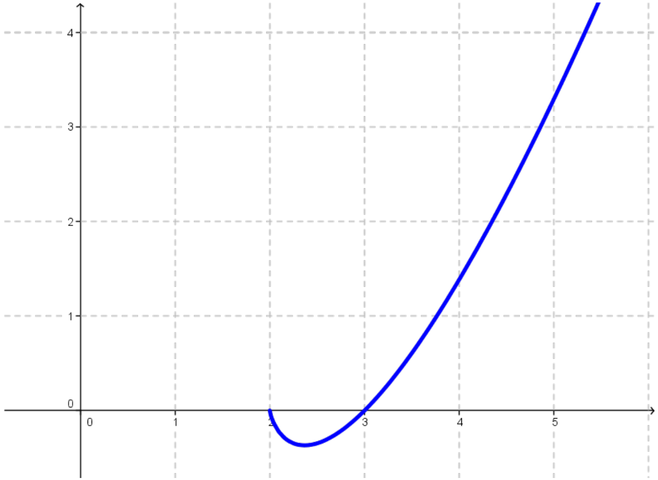
Curve sketching

1. Find the domain of f.
2. Find critical values of f and where f is increasing/decreasing.
3. Find possible inflection points of f and where f is concave up/concave down.
4. Find any vertical asymptotes.
5. Find any horizontal asymptotes.
6. Evaluate f at each critical point and possible point of inflections, and connect these points with curves of the proper concavity.

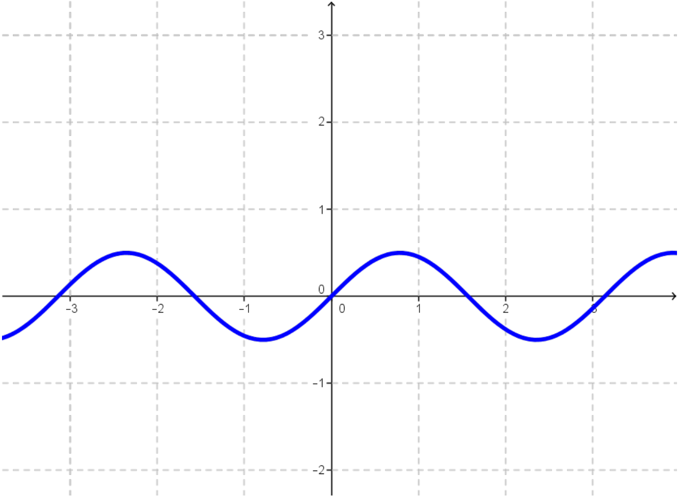
Example

1. $f(x) = -x^3 + 5x^2 - 3x + 2$
2. $f(x) = (x-2)\ln(x-2)$
3. $f(x) = \sin x \cos x$ on $[-\pi, \pi]$
4. $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 6x + 8}$

$$f(x) = -x^3 + 5x^2 - 3x + 2$$



$$f(x) = (x-2)\ln(x-2)$$



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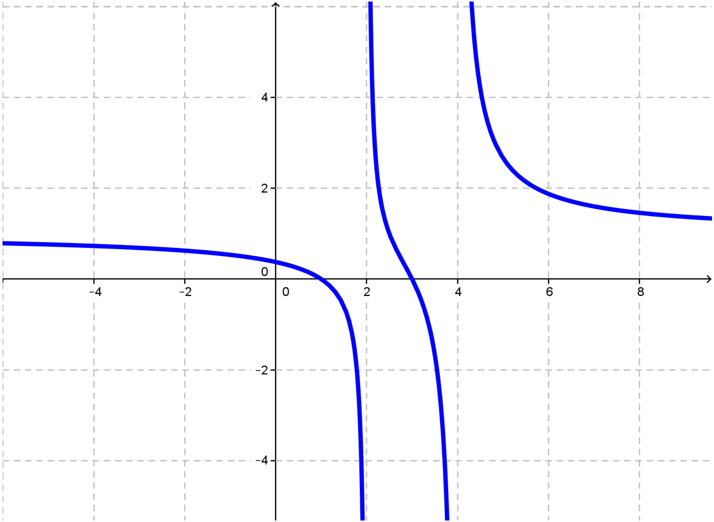

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 6x + 8}$$