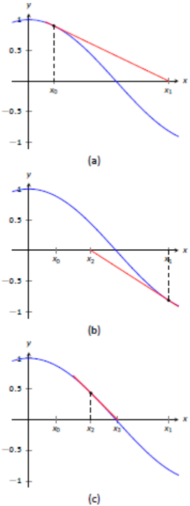
Idea

If x is sufficiently close to a root of $f(x)$, then the tangent line to the graph at $(x, f(x))$ will cross the x-axis at a point closer to the root than x.

Thus, iterating the procedure will "zero" in on the root.



Algebra

- Let x_0 be the initial x-value. The tangent line to $y = f(x)$ at the point $(x_0, f(x_0))$ has an equation

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- Let x_1 be the point where this line intersects the x-axis. Then

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- Now suppose we have computed $x_1, x_2, \ldots, x_n$. To get the next estimate x_{n+1} of the root, we compute

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Example

1. Find the real root of the function $f(x) = x^3 + x + 1$, accurate to two decimal places.
2. To two decimal places of accuracy, find the value of x where $\cos x = x$.

Just checking. . . .

1. Find the root of $f(x) = x + \ln x$, accurate to two decimal places.
2. Where is the function $f(x) = 2x^3 + x^2 - x + 3$ increasing, and where is it decreasing?
3. Find dy/dx if $x^{2/3} + y^{2/3} = 1$.
4. Find dy/dx if $y = \csc^{-1}(3x)$.
5. How close to 1 does x need to be in order that $\ln x$ be within 0.1 of $\ln 1 = 0$?

Method

1. Draw a picture and label it with variables and constants.

   Anything that varies in the problem should be a variable; only if a quantity remains constant throughout the problem can it be labeled as a constant.

2. Write down the rate(s) you know and the rate you want to find out *as derivatives*.

3. Find an equation that relates the variables involved in the known rate(s) and desired rate.

4. Apply d/dt to the above equation to differentiate with respect to time (using implicit differentiation), and plug in instantaneous values of variables and valus of known rates to solve for the desired rate.

Example

1. Water is being pumped into an empty spherical tank at 10 cubic feet per minute starting at time $t = 0$ minutes. The radius of the tank is 8 feet. At time $t = 2$ minutes, how fast is the depth of the water increasing? Note that the volume of a spherical cap of radius R and depth y is $V = (\pi/3)y^2(3R - y)$.

2. A radar station is tracking a plane. The plane is flying straight and level at 5000 feet and on a course that will take it directly over the radar dish. Its speed is 880 feet per second. How fast is the angle that the line connecting the dish to the plane makes with the ground changing when the plane is 10,000 feet away from the dish?

Example

1. The volume of a spherical balloon is increasing at 1/2 cubic feet per minute. How fast is the radius increasing when the volume is 1/4 cubic feet? How about the surface area?

Example

1. A pebble thrown into the pond produces circular ripples whose radii are growing at a rate of 5 in/s. At what rate is the circumference a ripple growing when the ripple has a radius of 10 in? At what rate is the area growing at that time?

2. A police officer is driving north at 30 mph on US 31 when she sees you cross the highway heading east on Port Sheldon Road. Suspecting that you might be speeding, she turns on her radar gun to get a reading. By using landmarks, she estimates that you and she are both 1/2 mile from the intersection when she takes a radar gun reading of 25 mph. Does she turn onto Port Sheldon and flip on her lights to pull you over, or were you observing the posted speed limit of 55 mph on Port Sheldon Road?

Just checking. . . .

1. Find $\lim_{x\to 2} \dfrac{\frac{3}{x} - \frac{3}{2}}{x-2}$.
2. Find the value of c that makes the function
$$f(x) = \left\{ \begin{array}{ll} 3x-2 & x \le 1 \\ 2x^2 + c & x < 1 \end{array} \right\}$$
continuous at $x = 1$. With this value of c, is f differentiable at $x = 1$?
3. Is there a value of m that makes the function
$$f(x) = \left\{ \begin{array}{ll} \sin(3x - \pi/6) & x < 0 \\ mx - 1/2 & \end{array} \right\}$$
continuous at $x = 0$? differentiable at $x = 0$?
4. Find dy/dx if $y = 3^{4x}$.
5. At what values of x, if any, does the function $f(x) = \frac{2x^2+6}{x-1}$ have a horizontal tangent? Give equations for any horizontal tangents, horizontal asymptotes, and vertical asymptotes.

Method

1. Draw a picture and label relevant variables and constants.
2. Identify the quantity to be minimized or maximized and the constraint.
3. Use the constraint to write the quantity to be optimized as a function of a single variable, and find the extreme values using the first derivative.

Example

1. Find the maximum product of two numbers that have a sum of 100.
2. Find the maximum sum of two numbers in $[0, 60]$ whose product is 100.
3. Find the maximum area of a right triangle whose hypotenuse has length 1.

Example

1. You are walking along the beach at Tunnel Park with your dog, who can run about 22 ft/s and swim about 1 ft/s. You throw a stick into the lake 20 feet down the shore line and 15 feet into the water.
   How far along the shore should your dog run before jumping into the lake in order to minimize the time it takes to get the stick?
2. The United States Postal Service charges more for boxes whose combined length and girth exceeds 108 inches. (The length of a package is the length of its longest side; the girth is the perimeter of the cross section.)
   What is the maximum volume of a package with a square cross section that does not exceed the 108-inch standard?

Example

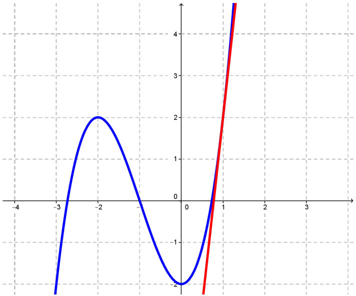
1. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.
2. A right triangle whose hypotenuse is $\sqrt{3}$ meters long is revolved about one of its legs to generate a right circular cone. Findthe radius, height , and volume of the cone of greatest volume that can be made this way.
3. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10. How about the same problem with an inscribed cone?

Just checking. . . .

1. Use an $\varepsilon - \delta$ argument to show that $\lim_{x \to 2} x^2 - 3 = 1$.
2. Use the definition of the derivative to show that $\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$. (Hint: $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$.)
3. Find $\lim_{x \to 0} \dfrac{\tan(2x)}{3x}$.
4. Find the maximum and minimum values of $f(x) = x^3 + 3x^2 - 2$ on $[-3, 2]$.
5. Find the value(s) of c where the instantaneous rate of change at c equals the average rate of change over $[-3, 2]$ for the function $f(x) = x^3 + 3x^2 - 2$.

Idea

Differentiable functions are virtually indistinguishable from the tangent line near the point of tangency. So, we can use points on the tangent line to approximate points on the curve near the point of tangency.



$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

Example

1. You have 12.2 feet of chicken wire. Approximate the area of the largest chicken pen you can make.
2. The radius of the earth is 3,959 miles. A fiber optics cable is laid around the equator. About how much extra cable needs to be rolled out in order to place it atop 30 foot poles.
3. The distance (in feet) a stone drops in t seconds is $d(t) = 16t^2$. You want to approximate the depth of an old well on your property, and you think you can measure time on your wrist watch accurately to about half a second. What is the propagated error if the time measurement is 2 seconds? 5 seconds?

Just checking. . . .

1. Approximate $\sqrt{9.4}$ using Newton's method.
2. Approximate $\sqrt{9.4}$ using differentials.
3. Approximate $e^{\pi/3}$.
4. A spherical balloon is inflated with helium flowing at a rate of 10 cm^3/min. How fast is the radius of the balloon increasing when the radius is 1 cm? 10 cm? 100 cm?
5. The strength S of a wooden beam is directly proportional to its cross sectional width and the square of its height. What are the dimensions of the strongest beam that can be cut from a log whose diameter is 12 inches?