Definition

Let $f(x)$ be a function. An **antiderivative** of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x)$$

The set of all antiderivatives of $f(x)$ is the **indefinite integral** of f, denoted by

$$\int f(x)\,dx$$

Theorem

If $f'(x) = 0$ for all x on an interval I, then f is constant on I.

Proof.

Theorem

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C$$

Proof.

Remark

Thus, any two antiderivatives of a function differ only by an additive constant, and so we commonly write

$$\int f(x)\, dx = F(x) + C$$

Example

1. $\int \sin x \, dx =$ ______________________
2. $\int e^x \, dx =$ ______________________
3. $\int \frac{1}{x} \, dx =$ ______________________
4. $\int 3x^2 + 4x - 1 \, dx =$ ______________________
5. $\int 0 \, dx =$ ______________________
6. $\int \, dx =$ ______________________

Example

1. $\int \sin x \, dx =$

   $-\cos x + C$

2. $\int e^x \, dx =$

   $e^x + C$

3. $\int \frac{1}{x} \, dx =$

   $\ln x + C$[1]

4. $\int 3x^2 + 4x - 1 \, dx =$

   $x^2 + 2x^2 - x + C$

5. $\int 0 \, dx =$

   C

6. $\int \, dx =$

   $x + C$

Remark

By checking our answers, we see that differentiating "undoes" integrating:

$$\frac{d}{dx}\left(\int f(x) \, dx\right) = f(x)$$

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[1] To make the domains agree, we should write $\int (1/x) \, dx = \ln|x| + C$.

Theorem 35 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\, dx = c \cdot \int f(x)\, dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\, dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\, dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\, dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\, dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\, dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\, dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\, dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\, dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\, dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\, dx = \ln\lvert x\rvert + C$

Remark

1. From this table we can see that differentiation and integration are inverse operations: each one "undoes" the other. Inverse operations do the opposite things in opposite order.
2. Antidifferentiation produces a family of functions: if F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C$$

So antiderivatives are determined up to an additive constant. We can determine the additive constant when we have some additional information about the antiderivative.

Initial value problems

Example

1. Find a solution to $\dfrac{dy}{dx} = x^3 + \dfrac{3}{x}$ if $y = 6.25$ when $x = 1$.
2. If $f''(x) = 24x^2 + 12x - 8$ and $f(0) = 5$ and $f(1) = 1$, find $f(x)$.
3. A car braked with a constant deceleration of 40 feet per second squared, producing skid marks measuring 160 feet before coming to a stop. How fast was the car traveling when the brakes were first applied?
4. A car is traveling 80 feet per second (approximately 55 miles per hour) when the brakes are fully applied, producing a constant deceleration of 40 feet per second squared. What is the distance traveled between when the brakes are first applied and when the car comes to a stop?

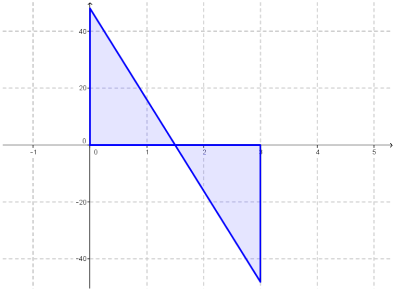
Just checking. . . .

1. Find $\int (x^2 + 3)(x - 2)\, dx$.
2. Given $y = x^2 e^x \cos x$, find dy.
3. Find dy/dx if $x^3 + xy - \cos y = 5$.
4. Find the area of the largest rectangle that can be placed between the x-axis and the parabola $y = 9 - x^2$.
5. Find $f(x)$ if $f''(x) = x$ and $f(0) = 1$ and $f(2) = 3$.

Example

The velocity of a baseball moving straight up and down under the acceleration of gravity is $v(t) = -32t + 48$, where time t is given in seconds and velocity v is in ft/s. When $t = 0$, the baseball had a height of 0 ft.

1. What was the initial velocity of the baseball?
2. What was the maximum height of the baseball?
3. What was the height of the baseball at time $t = 2$?



Definition

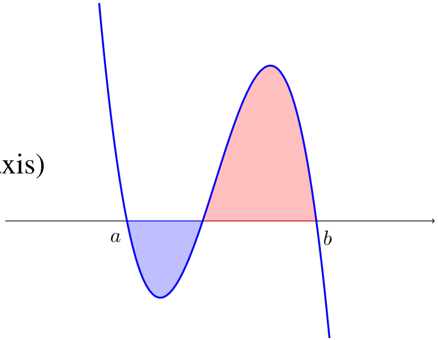
Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area** between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is

(area above x-axis)−(area below x-axis)

The **definite integral**

$$\int_a^b f(x)\,dx$$

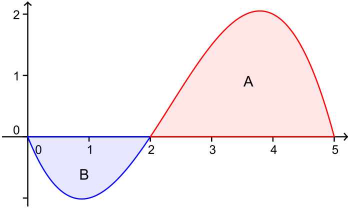
computes the total signed area.



$$\int_a^b f(x)\,dx = \text{red} - \text{blue}$$

Example

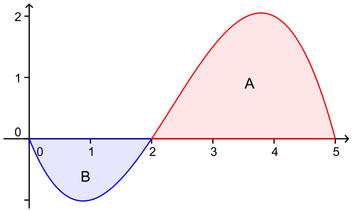
Suppose the area of region A is 4 and the area of region B is 1.



1. $\int_0^2 f(x)\,dx =$ ____________
2. $\int_3^3 f(x)\,dx =$ ____________
3. $\int_2^5 f(x)\,dx =$ ____________
4. $\int_0^5 f(x)\,dx =$ ____________
5. $\int_0^2 |f(x)|\,dx =$ ____________
6. $\int_2^5 -f(x)\,dx =$ ____________

Example

Suppose the area of region A is 4 and the area of region B is 1.



1. $\int_0^2 f(x)\,dx = -1$
2. $\int_3^3 f(x)\,dx = 0$
3. $\int_2^5 f(x)\,dx = 4$
4. $\int_0^5 f(x)\,dx = 3$
5. $\int_0^2 |f(x)|\,dx = 1$
6. $\int_2^5 -f(x)\,dx = -4$

Example

Evaluate each integral by interpreting it in terms of signed area.

1. $\int_1^3 7\,dx =$ ______________
2. $\int_1^3 (1+2x)\,dx =$ __________
3. $\int_{-2}^3 |x-2|\,dx =$ __________
4. $\int_{-2}^3 |x|-2\,dx =$ __________
5. $\int_{-5}^5 \sqrt{25-x^2}\,dx =$ ________
6. $\int_0^5 -\sqrt{25-x^2}\,dx =$ ______

Example

Evaluate each integral by interpreting it in terms of signed area.

1. $\int_1^3 7\,dx = 14$
2. $\int_1^3 (1+2x)\,dx = 10$
3. $\int_{-2}^3 |x-2|\,dx = 17/2$
4. $\int_{-2}^3 |x| - 2\,dx = -7/2$
5. $\int_{-5}^5 \sqrt{25-x^2}\,dx = 25\pi/2$
6. $\int_0^5 -\sqrt{25-x^2}\,dx = -25\pi/4$

Theorem

Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant.

1. $$\int_a^a f(x)\,dx = 0$$
2. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$
3. $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
4. $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$
5. $$\int_a^b k \cdot f(x)\,dx = f \cdot \int_a^b f(x)\,dx$$

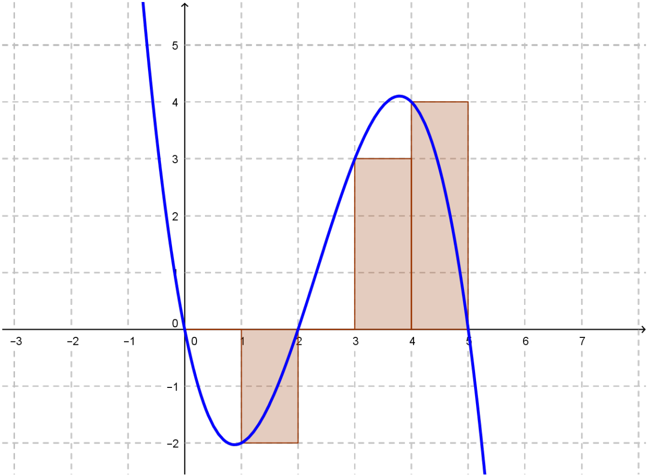
Just checking. . . .

Suppose

- $\int_0^2 f(x)\,dx = 5$
- $\int_0^3 f(x)\,dx = 7$
- $\int_0^2 g(x)\,dx = -3$
- $\int_2^3 g(x)\,dx = 5$

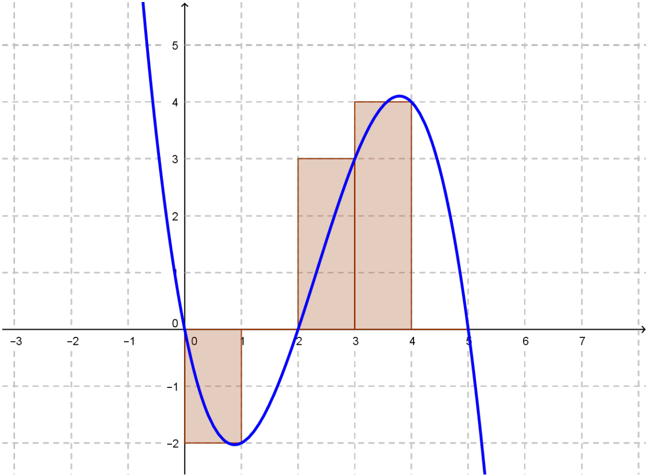
1. Find $\int_0^2 f(x) + g(x)\,dx$
2. Find $\int_2^3 3f(x) - 2g(x)\,dx$
3. Find $\int_0^3 4g(x) - x + 1\,dx$
4. Find values for a and b such that $\int_0^3 af(x) + bg(x)\,dx = 0$
5. Find $\int_{-\pi}^{\pi} \sin x\,dx$

Idea: add up areas of rectangles to approximate



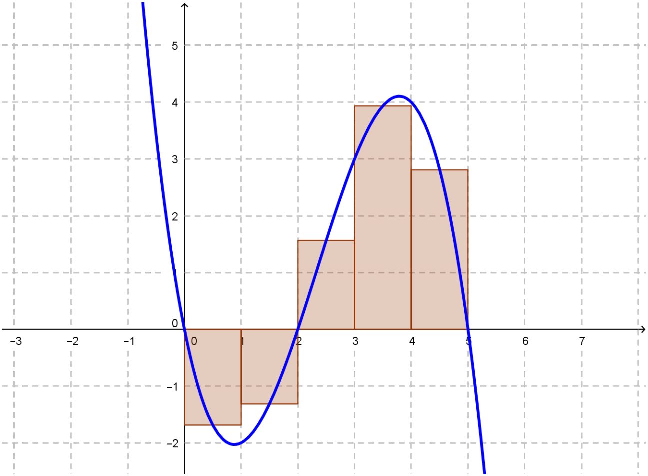
5 rectangles, left endpoints

Idea: add up areas of rectangles to approximate



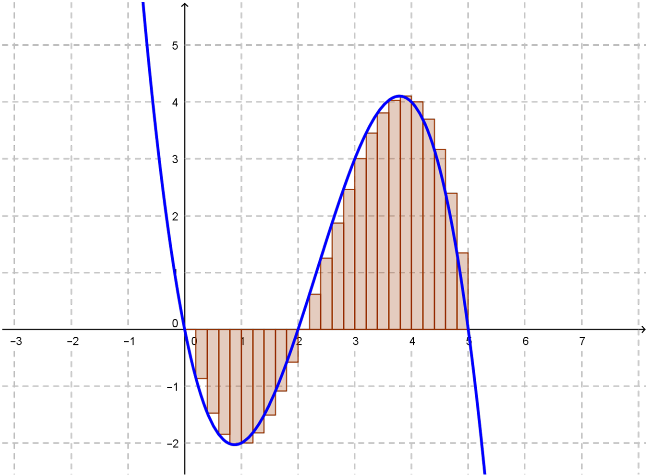
5 rectangles, right endpoints

Idea: add up areas of rectangles to approximate



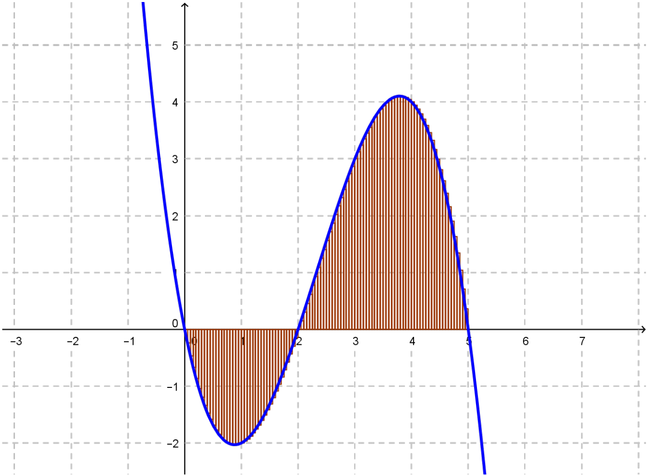
5 rectangles, midpoints

Idea: add up areas of rectangles to approximate



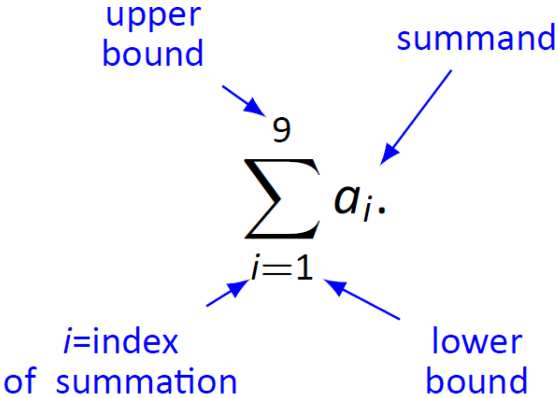
25 rectangles, left endpoints

Idea: add up areas of rectangles to approximate



100 rectangles, left endpoints

Sigma notation



$$\sum_{i=1}^{9} a_i = a_1 + a_2 + a_3 + \cdots + a_9$$

Sigma notation

Example

Expand the following summations and evaluate the sums.

1. $\sum_{k=2}^{5} 3k =$ ______________
2. $\sum_{j=-3}^{1} j^2 =$ ______________
3. $\sum_{i=0}^{4} i/3 =$ ______________
4. $\sum_{n=1}^{4} n^2 - n =$ ______________

Sigma notation

Example

Expand the following summations and evaluate the sums.

1. $\sum_{k=2}^{5} 3k = 42$
2. $\sum_{j=-3}^{1} j^2 = 15$
3. $\sum_{i=0}^{4} i/3 = 10/3$
4. $\sum_{n=1}^{4} n^2 - n = 20$

Sigma notation

Theorem

Formal properties

1. $$\sum_{i=m}^{n}(a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$$
2. $$\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$$
3. $$\sum_{i=m}^{k} a_i + \sum_{i=k+1}^{n} a_i = \sum_{i=m}^{n} a_i$$

Common sums

1. $$\sum_{i=1}^{n} 1 = n$$
2. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
3. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
4. $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Method

Let n be a fixed positive integer.

1. Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.
2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.
3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.
4. Add the areas of all n rectangles to compute the n^{th} Riemann sum:

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example

Compute $\int_1^3 x^2 \, dx$.

1. Divide $[a, b]$ into n subintervals of width $\Delta x = \dfrac{b-a}{n}$.

   $\Delta x =$ ______________________________

2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.

   $x_i =$ ______________________________

3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.

   $f(x_i) =$ ______________________________

4. Add the areas of all n rectangles to compute the n^{th} Riemann sum R_n:

   $$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

   $$\int_1^3 f(x)\, dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Example

Compute $\int_1^3 x^2\,dx$.

1. Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

   $\Delta x = 2/n$

2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.

   $x_i =$ ______________________

3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.

   $f(x_i) =$ ______________________

4. Add the areas of all n rectangles to compute the n^{th} Riemann sum R_n:

   $R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x =$ ______________________

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

   $\int_1^3 f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x =$ ______________________

Example

Compute $\int_1^3 x^2\,dx$.

1. Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

   $\Delta x = 2/n$

2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.

   $x_i = 1 + 2i/n$

3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.

   $f(x_i) =$ ____________________

4. Add the areas of all n rectangles to compute the n^{th} Riemann sum R_n:

   $$R_n = \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

   $$\int_1^3 f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \text{\_\_\_\_\_\_\_\_\_\_\_\_}$$

Example

Compute $\int_1^3 x^2\,dx$.

1. Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

   $\Delta x = 2/n$

2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.

   $x_i = 1 + 2i/n$

3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.

   $f(x_i) = 1 + 4i/n + 4i^2/n^2$

4. Add the areas of all n rectangles to compute the n^{th} Riemann sum R_n:

   $$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

   $$\int_1^3 f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Example

Compute $\int_1^3 x^2\,dx$.

1. Divide $[a,b]$ into n subintervals of width $\Delta x = \dfrac{b-a}{n}$.

   $\Delta x = 2/n$

2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.

   $x_i = 1 + 2i/n$

3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.

   $f(x_i) = 1 + 4i/n + 4i^2/n^2$

4. Add the areas of all n rectangles to compute the n^{th} Riemann sum R_n:

   $$R_n = \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8i^2}{n^3}\sum_{i=1}^{n} i^2 = 2 + 4\left(1+\frac{1}{n}\right) + \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

   $$\int_1^3 f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \underline{\hspace{8cm}}$$

Example

Compute $\int_1^3 x^2\,dx$.

1. Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

   $\Delta x = 2/n$

2. Find the right endpoint of the i^{th} subinterval $x_i = a + i\Delta x$.

   $x_i = 1 + 2i/n$

3. Use the right endpoint to compute the height of the i^{th} rectangle $f(x_i)$.

   $f(x_i) = 1 + 4i/n + 4i^2/n^2$

4. Add the areas of all n rectangles to compute the n^{th} Riemann sum R_n:

   $$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8i^2}{n^3}\sum_{i=1}^{n} i^2 = 2 + 4\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

5. Take the limit as $n \to \infty$ (so that $\Delta x \to 0$):

   $$\int_1^3 f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = 26/3$$

Example

1. Compute $\int_1^3 x^2 + 1\,dx$. (Hint: cheat!)

   $\int_1^3 x^2 + 1\,dx =$ ________________________________

2. Compute $\int_0^4 -x^2 + 4x\,dx$. (Hint: don't cheat!)

   $\int_0^4 4x - x^2\,dx =$ ________________________________

Remark

1. Compute $\int x^2 + 1\,dx = F(x)$ and evaluate $F(3)$ and $F(1)$.
2. Compute $\int 4x - x^2\,dx = F(x)$ and evaluate $F(4)$ and $F(0)$.

Example

1. Compute $\int_1^3 x^2 + 1\,dx$. (Hint: cheat!)

   $$\int_1^3 x^2 + 1\,dx = 26/3 + 2 = 32/3$$

2. Compute $\int_0^4 -x^2 + 4x\,dx$. (Hint: don't cheat!)

   $$\int_0^4 4x - x^2\,dx = \rule{10cm}{0.4pt}$$

Remark

1. Compute $\int x^2 + 1\,dx = F(x)$ and evaluate $F(3)$ and $F(1)$.
2. Compute $\int 4x - x^2\,dx = F(x)$ and evaluate $F(4)$ and $F(0)$.

Example

1. Compute $\int_1^3 x^2 + 1\,dx$. (Hint: cheat!)

   $$\int_1^3 x^2 + 1\,dx = 26/3 + 2 = 32/3$$

2. Compute $\int_0^4 4x - x^2\,dx$. (Hint: don't cheat!)

   $$\int_0^4 4x - x^2\,dx = 32/3$$

Remark

1. Compute $\int x^2 + 1\,dx = F(x)$ and evaluate $F(3)$ and $F(1)$.
2. Compute $\int 4x - x^2\,dx = F(x)$ and evaluate $F(4)$ and $F(0)$.

Just checking. . . .

1. Write $1 + 4 + 9 + \cdots + 400$ in sigma notation.
2. Evaluate $\displaystyle\sum_{i=15}^{71} i$.
3. Find an antiderivative of $1/\sqrt{x}$.
4. Approximate $\displaystyle\int_1^3 \ln x\, dx$ using four rectangles of equal width and right endpoints.
5. Compute $\displaystyle\int_{-10}^{10} 5 - x\, dx$ using Riemann sums and check your answer using geometry.

Definition

Given a closed interval $[a, b]$. A *partition* of $[a, b]$ is a set of numbers $\mathcal{P} = \{x_0, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$ that divides $[a, b]$ into n subintervals. Suppose that all subintervals are of equal length, $\Delta x = (b - a)/n$. In each subinterval $[x_{k-1}, x_k]$, pick a number x_k^*. A function f is *integrable* on $[a, b]$ if

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

exists independent of the choices of x_k^* in $[x_{k-1}, x_k]$. In this case, the *definite integral of f from a to b*, written

$$\int_a^b f(x)\, dx$$

is the value of that limit.

The mean value theorem of integration

Theorem

Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Definition

The **average value of** f **on** $[a, b]$ is $f(c)$, where c is the value guaranteed by the mean value theorem of integration. That is,

$$\text{Average value of } f \text{ on } [a,b] = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Theorem

Part 1 of the FTC

Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt$. Then F is a differentiable function on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Remark

1. $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f.
2. The rate at which area is being added under the graph of $y = f(x)$ at any point x is equal to the value of the function $f(x)$.

Example

1. $\dfrac{d}{dx}\displaystyle\int_3^x \sin t^2\,dt =$

   $=$ ________________________

2. $\dfrac{d}{dx}\displaystyle\int_x^7 e^{t^3}\,dt =$

   $=$ ________________________

3. $\dfrac{d}{dx}\displaystyle\int_{-1}^{x^2} \ln t\,dt =$

   $=$ ________________________

4. $\dfrac{d}{dx}\displaystyle\int_x^{x^2} t^2 + 3t\,dt =$

   $=$ ________________________

Example

1. $\frac{d}{dx}\int_3^x \sin t^2\,dt =$

   $= \sin x^2$

2. $\frac{d}{dx}\int_x^7 e^{t^3}\,dt =$

   $= -e^{x^3}$

3. $\frac{d}{dx}\int_{-1}^{x^2} \ln t\,dt =$

   $= 2x\ln(x^2) = 4x\ln x$

4. $\frac{d}{dx}\int_x^{x^2} t^2 + 3t\,dt =$

   $= 2x^5 + 6x^3 - x^2 - 3x$

Theorem

Part 2 of the FTC

Let f be continuous on $[a, b]$, and let F be any antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example

1. $\int_0^4 4x - x^2\,dx =$ __________
2. $\int_0^{\pi/3} \sin x\,dx =$ __________
3. $\int_0^2 3^x\,dx =$ __________
4. $\int_0^2 x(x-1)(x-2)\,dx =$ ____

Theorem

Part 2 of the FTC

Let f be continuous on $[a, b]$, and let F be any antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example

1. $\int_0^4 4x - x^2\,dx = 32/3$
2. $\int_0^{\pi/3} \sin x\,dx = 1/2$
3. $\int_0^2 3^x\,dx = 8/\ln(3)$
4. $\int_0^2 x(x-1)(x-2)\,dx = 0$

Motion

Remark

Recall that $\frac{d}{dt}(s) = v$ and that $\frac{d}{dt}(v) = a$. So by the FTC

$$\int_a^b v(t)\,dt = s(t)\Big|_a^b = s(b) - s(a)$$

and

$$\int_a^b a(t)\,dt = v(t)\Big|_a^b = v(b) - v(a)$$

Example

Suppose an object moves in a straight line with velocity $v(t) = t^2 - 2t$ m/s.

1. When is the object moving forward? backward?
2. What is the object's displacement over $[1, 3]$?
3. What is the distance the object travels over $[1, 3]$?

Area between curves

Theorem

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$, where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$ is

$$\int_a^b f(x) - g(x)\, dx$$

Example

1. Find the area enclosed by $y = x^2 - 2x + 5$ and $y = 5x - 5$.
2. Find the area of one lobe bounded by the curves $y = \sin x$ and $y = \cos x$.

The mean value theorem of integration

Theorem

Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

Definition

The **average value of f on** $[a, b]$ is $f(c)$, where c is the value guaranteed by the mean value theorem of integration. That is,

$$\text{Average value of } f \text{ on } [a, b] = f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

Example

An object moves along a straight line with velocity $v(t) = t^2 - 2t$ m/s. What is its average velocity over $[0, 3]$?

Just checking. . . .

1. Find $\dfrac{d}{dx}\displaystyle\int_3^{x^3} \sin t^2 \, dt$.
2. Find the area between $y = x$ and $y = x^2/4$.
3. Find a value c guaranteed by the mean value theorem for integration for $\displaystyle\int_0^2 x^2 \, dx$.
4. Find $\displaystyle\int_1^e \frac{dx}{x}$.
5. Find the area bounded by $y = 3x^2 - 3$ and the x-axis over the interval $[-2, 2]$.

Example

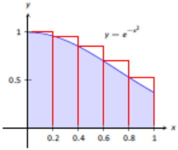
Estimate $\int_0^1 e^{-x^2}\,dx$ using 5 equally spaced subintervals and

1. the left-endpoint method
2. the right-endpoint method
3. trapezoids

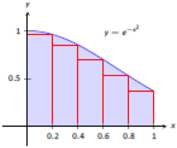
x_i	$f(x_i) = e^{-x_i^2}$
0	1
0.2	0.9608
0.4	0.8521
0.6	0.6977
0.8	0.5273
1	0.3679

Remark

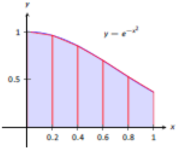
- The left- and right-endpoint methods approximate the function with a *constant* function over each subinterval.
- The trapezoid method approximates the function with a *linear* function over each subinterval.
- The next method, known as Simpson's method, approximates the function with a *quadratic* function over each subinterval.



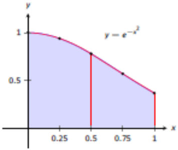
Left



Right



Trapezoid



Simpson's

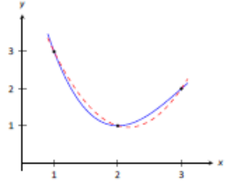
Simpson's method

Key ideas

1. There is a unique parabola that goes through any three noncollinear points.
2. If $f(x)$ is the parabola going through the noncollinear points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), where $x_1 < x_2 < x_3$ are equally spaced, then

$$\int_{x_1}^{x_3} f(x)\, dx = \frac{x_3 - x_1}{6}\,(y_1 + 4y_2 + y_3)$$

3. Simpson's method requires an even number of subintervals since two subintervals are used to create the parabolic approximation of the function.



Errors

Theorem

1. Let E_T be the error in approximating $\int_a^b f(x)\,dx$ using the Trapezoidal Rule. If f has a continuous 2^{nd} derivative on $[a, b]$ and M is any upper bound of $|f''(x)|$ on $[a, b]$, then

$$E_T \leq \frac{(b-a)^3}{12n^2}M$$

2. Let E_S be the error in approximating $\int_a^b f(x)\,dx$ using Simpson's Rule. If f has a continuous 4^{th} derivative on $[a, b]$ and M is any upper bound of $|f^{(4)}(x)|$ on $[a, b]$, then

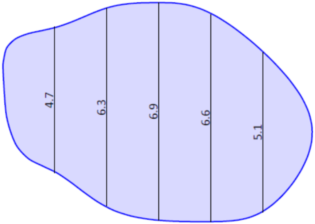
$$E_S \leq \frac{(b-a)^5}{180n^4}M$$

Remark

1. The larger the interval, the larger the error.
2. The error shrinks as more subintervals are used (i.e. as n gets larger)
3. The error in Simpson' rule has a term relating to the 4^{th} derivative of f. Therefore, the error in approximating a definite integral of a cubic polynomial using Simpson's Rule is 0 – Simpson's Rule computes the exact answer!

Just checking. . . .

1. Approximate the area of the region if
   a. the measurements are in centimeters, taken in 1 cm increments.
   b. the measurements are in hundreds of yards, taken in 100 yd increments.



2. Find n such that the error in approximating the given definite integral is less than $0.0001 = 10^{-4}$ when using
   a. the Trapezoidal Rule
   b. Simpson's Rule

3. Approximate the definite integral $\int_1^4 \ln x \, dx$ using $n = 6$ with
   a. the Trapezoidal Rule
   b. Simpson's Rule