Example

1. Find $\dfrac{d}{dx}\left(x^2 + 3x - 5\right)^{10}$.
2. Find $\displaystyle\int (20x + 30)(x^2 + 3x - 5)^9 \, dx$.

Remark

More generally $\dfrac{d}{dx}(F(g(x))) = F'(g(x))g'(x)$ and so

$$\int F'(g(x))g'(x)\, dx = F(g(x)) + C.$$

Letting $u = g(x)$, so that $du = g'(x)\, dx$, we have

$$\int F'(u)\, du = F(u) + C = F(g(x)) + C.$$

Theorem

Let F and g be differentiable functions, where the range of g is contained in the domain of F. Then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

If $u = g(x)$ so that $du = g'(x)\,dx$, then

$$\int F'(g(x))g'(x)\,dx = \int F'(u)\,du = F(u) + C = F(g(x)) + C$$

Example

1. $\int xe^{x^2}\,dx =$ ______________________

2. $\int x^2 \cos x^3\,dx =$ ______________________

3. $\int \sin x \cos x\,dx =$ ______________________

4. $\int \frac{e^x + 1}{e^x}\,dx =$ ______________________

5. $\int \frac{3x^2 - 5x + 7}{x + 1}\,dx =$ ______________________

6. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx =$ ______________________

Example

1. $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$
2. $\int x^2 \cos x^3\,dx = \frac{1}{3}\sin x^3 + C$
3. $\int \sin x \cos x\,dx = -\frac{1}{2}\cos^2 x + C$
4. $\int \frac{e^x + 1}{e^x}\,dx = x - e^{-x} + C$
5. $\int \frac{3x^2 - 5x + 7}{x + 1}\,dx = \frac{3}{2}x^2 - 8x + 15\ln|x + 1| + C$
6. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\sin\sqrt{x} + C$

Example

1. $\displaystyle\int \frac{x^3 - x}{\sqrt{x}}\, dx =$ ______________________
2. $\displaystyle\int \frac{\ln x}{x}\, dx =$ ______________________
3. $\displaystyle\int \frac{\ln \sqrt{x}}{x}\, dx =$ ______________________
4. $\displaystyle\int \tan x\, dx =$ ______________________
5. $\displaystyle\int x\sqrt{1 - x^2}\, dx =$ ______________________
6. $\displaystyle\int x\sqrt{1 - x}\, dx =$ ______________________

Example

1. $\displaystyle\int \frac{x^3 - x}{\sqrt{x}}\,dx = \frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + C$
2. $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$
3. $\displaystyle\int \frac{\ln \sqrt{x}}{x}\,dx = \frac{1}{4}(\ln x)^2 + C$
4. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
5. $\displaystyle\int x\sqrt{1 - x^2}\,dx = -\frac{1}{3}(1 - x^2)^{3/2} + C$
6. $\displaystyle\int x\sqrt{1 - x}\,dx = \frac{2}{5}(1 - x)^{5/2} - \frac{2}{3}(1 - x)^{3/2} + C$

Trigonometric integrals

Theorem

1. $\displaystyle\int \cos x\, dx = \sin x + C$
2. $\displaystyle\int \sin x\, dx = -\cos x + C$
3. $\displaystyle\int \tan x\, dx = -\ln|\cos x| + C$
4. $\displaystyle\int \cot x\, dx = \ln|\sin x| + C$
5. $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$
6. $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

Trigonometric integrals

Theorem

Angle Addition Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Example

1. $\sin 2\theta =$ ____________________
2. $\cos 2\theta =$ ____________________

Example

1. $\sin^2(\theta/2) =$ ________________
2. $\cos^2(\theta/2) =$ ________________

Trigonometric integrals

Theorem

Angle Addition Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Example

1. $\sin 2\theta = 2\sin\theta\cos\theta$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Example

1. $\sin^2(\theta/2) =$ ________________
2. $\cos^2(\theta/2) =$ ________________

Trigonometric integrals

Theorem

Angle Addition Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Example

1. $\sin 2\theta = 2\sin\theta\cos\theta$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Example

1. $\sin^2(\theta/2) = \dfrac{1 - \cos\theta}{2}$
2. $\cos^2(\theta/2) = \dfrac{1 + \cos\theta}{2}$

Trigonometric integrals

Example

1. $\int \cos^2 x \, dx =$ ______________________________
2. $\int \sin^2 x \, dx =$ ______________________________
3. $\int \tan^2 x \, dx =$ ______________________________
4. $\int \cos^4 x \, dx =$ ______________________________
5. $\int \cos^3 x \, dx =$ ______________________________
6. $\int \tan^3 x \, dx =$ ______________________________

Trigonometric integrals

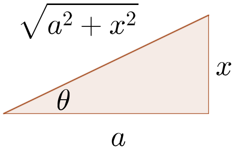
Example

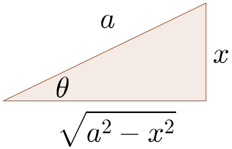
1. $$\int \cos^2 x\,dx = \frac{1}{2}\sin x\cos x + \frac{1}{2}x + C$$
2. $$\int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C$$
3. $$\int \tan^2 x\,dx = x - \tan x + C$$
4. $$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C$$
5. $$\int \cos^3 x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C$$
6. $$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \frac{1}{2}\ln(1 + \tan^2 x) + C$$

6.1 Substitution

Trigonometric substitution

A sum or difference of squares underneath a square root sign may be interpreted as one side of a right triangle.





$$x = a \tan \theta$$
$$dx = a \sec^2 \theta \, d\theta$$

$$x = a \sin \theta$$
$$dx = a \cos \theta \, d\theta$$

By making an appropriate substitution, we can convert integrals involving sums or differences of squares underneath a square root into trigonometric integrals.

Trigonometric substitution

Example

1. $$\int \frac{dx}{9+x^2} = \underline{\hspace{10cm}}$$
2. $$\int \frac{dx}{\sqrt{4-x^2}} = \underline{\hspace{10cm}}$$
3. $$\int \frac{9-x}{\sqrt{9-x^2}}\, dx = \underline{\hspace{10cm}}$$
4. $$\int \frac{x^2\, dx}{\sqrt{9-x^2}} = \underline{\hspace{10cm}}$$
5. $$\int \frac{1}{x^2-4x+13}\, dx = \underline{\hspace{10cm}}$$

Trigonometric substitution

Example

1. $$\int \frac{dx}{9+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$
2. $$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin(x/2) + C$$
3. $$\int \frac{9-x}{\sqrt{9-x^2}}\,dx = 9\arcsin(x/3) + \sqrt{9-x^2} + C$$
4. $$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$$
5. $$\int \frac{1}{x^2-4x+13}\,dx = \frac{1}{3}\arctan\left(\frac{x-2}{3}\right) + C$$

Substitution in definite integrals

Theorem

Let F and g be differentiable functions, where the range of g is contained in the domain of F. Then

$$\int_a^b F'(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} F'(u)\,du$$

Remark

So, when evaluating a definite integral where a substitution has been made, one can either

- convert the antiderivative to a function of x and use the original x-limits, or
- leave the antiderivative as a function of u and convert the x-limits to u-limits.

Substitution in definite integrals

Example

1. $\displaystyle\int_2^7 \sin(2x+3)\,dx =$ ____________________
2. $\displaystyle\int_0^{\pi/3} \sin^2 x \cos x\,dx =$ ____________________
3. $\displaystyle\int_1^9 \frac{\ln x^2}{x}\,dx =$ ____________________
4. $\displaystyle\int_1^9 \frac{(\ln x)^2}{x}\,dx =$ ____________________
5. $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}\,dx =$ ____________________
6. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx =$ ____________________

Substitution in definite integrals

Example

1. $\int_2^7 \sin(2x+3)\,dx = \frac{1}{2}\cos 7 - \frac{1}{2}\cos 17 \approx 0.5145$
2. $\int_0^{\pi/3} \sin^2 x \cos x\,dx = \sqrt{3}/8$
3. $\int_1^9 \frac{\ln x^2}{x}\,dx = 4(\ln 3)^2$
4. $\int_1^9 \frac{(\ln x)^2}{x}\,dx = \frac{8}{3}(\ln 3)^3$
5. $\int_{-1}^1 \frac{1}{1+x^2}\,dx = \pi/2$
6. $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \pi/6$