MATH 132 Worksheet 3 Integration: Partial Fractions

1. Set up the partial fraction decomposition (but do not solve!)

(a) $\dfrac{3}{x^2-4}$

(b) $\dfrac{2}{(x^2-4)(x^2-9)}$

(c) $\dfrac{1}{(x^2-4)(x-4)^2}$

(d) $\dfrac{2x}{(x^2-4)(x^2+4)}$

(e) $\dfrac{x-3}{(x^2-3x+2)(x-2)^2}$

(f) $\dfrac{5}{x^4(x^2-1)^2(x^2+1)^3}$

2. Find the partial fraction decomposition (i.e. set them up *and* solve them!)

(a) $\dfrac{3}{x^2-4}$

(b) $\dfrac{x^2+4x+1}{(x-1)(x+1)(x+3)}$

(c) $\dfrac{-2x+4}{(x^2+1)(x-1)^2}$

3. Integrate! (Use the results from above)

(a) $\displaystyle\int \frac{3}{x^2-4}\,dx$

(b) $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

(c) $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

4. Integrate!

(a) $\displaystyle\int \frac{1}{x^2+9}\,dx$

(b) $\displaystyle\int \sin^3(x)\,dx$

(c) $\displaystyle\int \frac{3}{\sqrt{9-x^2}}\,dx$

(d) $\displaystyle\int \sqrt{1-x^2}\,dx$

(e) $\displaystyle\int \cos^2(x)\,dx$

(f) $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

(g) $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$

(h) $\displaystyle\int \frac{3x^3}{x^2-1}\,dx$

(i) $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$

(j) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

do in 2 ways?

a bit involved/tricky

(k) $\displaystyle\int \frac{2}{\sqrt{25-4x^2}}\,dx$

(l) $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx$ pretty tricky

(m) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$

(n) $\displaystyle\int \cos(2x)\sin^9(2x)\,dx$

(o) $\displaystyle\int 3xe^x\,dx$

① ⓐ $= \frac{A}{x-2} + \frac{B}{x+2}$

ⓑ $= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-3} + \frac{D}{x+3}$

ⓒ $= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-4} + \frac{D}{(x-4)^2}$

ⓓ $= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$

ⓔ $= \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$

ⓕ $= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x-1} + \frac{F}{(x-1)^2} + \frac{G}{x+1} + \frac{H}{(x+1)^2} + \frac{Ix+J}{x^2+1} + \frac{Kx+L}{(x^2+1)^2} + \frac{Mx+N}{(x^2+1)^3}$

② ⓐ $\frac{3}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$

$3 = A(x+2) + B(x-2) = (A+B)x + 2A - 2B$

$A+B=0$, $2A-2B=3$, $B=-A$, $2A+2A=3$, $A=3/4$, $B=-3/4$

ⓑ $\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$
$$= A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)$$
$$= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C)$$

① $A+B+C=1$

② $4A+2B=4 \Rightarrow 2A+B=2$, $B=2-2A$.

③ $3A-3B-C=1$

①→ $A+2-2A+C=1$, $-A+C=-1$ $C=A-1$

③→ $3A-3(2-2A)-C=1$ $9A-C=7$

$9A-(A-1)=7$, $8A=6$, $\boxed{A=3/4}$

$\boxed{B = 2-2\cdot\frac{3}{4} = \frac{1}{2}}$ $\boxed{C=\frac{1}{4}}$

② ⓒ $$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$= (Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$$= Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 - Cx^2 + Cx - C + Dx^2 + D$$

$A + C = 0$ $\boxed{C = -A}$

$-2A + B - C + D = 0$

$A - 2B + C = -2$

$B - C + D = 4$

$-2A + B + A + D = 0$ $\quad -A + B + D = 0$ $\quad -A + D = -1$ $\quad D = A - 1$

$A - 2B - A = -2$ $\quad \boxed{B = 1}$ $\quad A + A - 1 = 3$ $\quad \boxed{A = 2}$

$B + A + D = 4$ $\quad A + D = 3$ $\quad \boxed{D = 1}$

$\boxed{C = -2}$

---

③ ⓐ $$\int \frac{3}{x^2-4}\,dx = \int \frac{3}{4}\frac{1}{x-2}\,dx - \frac{3}{4}\int \frac{1}{x+2}\,dx = \boxed{\frac{3}{4}\ln|x-2| - \frac{3}{4}\ln|x+2| + C}$$

ⓑ $$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} + \frac{1}{4}\frac{1}{x+3}\,dx$$

$$= \boxed{\frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x+3| + C}$$

ⓒ $$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int 2\frac{x}{x^2+1} + \frac{1}{x^2+1} - 2\frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \boxed{\ln|x^2+1| + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C}$$

$\sin^2 x + \cos^2 x = 1 \quad \tan^2 x + 1 = \sec^2 x$

4 (a) $\boxed{\frac{1}{3}\arctan(x/3)+C}$

(b) $= \int (1-\cos^2 x)\sin x\,dx = \boxed{-\cos x + \frac{1}{3}\cos^3 x + C}$

(c) $\int \frac{3}{\sqrt{9-x^2}}\,dx = \int \frac{9\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{9\cos\theta\,d\theta}{3\cos\theta} = \int 3\,d\theta = 3\theta + C$

$x = 3\sin\theta \Rightarrow \theta = \arcsin\frac{x}{3} = \boxed{3\arcsin(x/3)+C}$

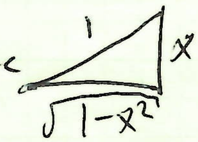
$dx = 3\cos\theta\,d\theta$

(d) $\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$

$x = \sin\theta$

$du = \cos\theta\,d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\arcsin\theta + \frac{1}{2}\sin\theta\cos\theta + C$



$= \boxed{\frac{1}{2}\arcsin\theta + \frac{1}{2}x\sqrt{1-x^2} + C}$

(e) $\int \cos^2(x)\,dx = \boxed{\frac{1}{2}x + \frac{1}{4}\sin 2x + C}$ (just did it in d)

(f) This is 3c

(g) $6x+7 = A(x+2)+B,\ A=6,\ 2A+B=7,\ B=-5$

$\int \frac{6}{x+2} + \frac{-5}{(x+2)^2}\,dx = \boxed{6\ln|x+2| + \frac{5}{x+2} + C}$

(h) $\int \frac{3x^3}{x^2-1}\,dx = \frac{3}{2}\int \frac{(u-1)\,du}{u} = \frac{3}{2}\int 1 - \frac{1}{u}\,du = \frac{3}{2}u - \ln|u| + C$

$u = x^2-1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \boxed{\frac{3}{2}(x^2-1) - \ln|x^2-1| + C}$

(i) $\int \frac{1}{x(x^2+1)^2}\,dx = \int \frac{\sec^2\theta\,d\theta}{\tan\theta(\tan^2\theta+1)^2} = \int \frac{\sec^2\theta\,d\theta}{\tan\theta\sec^4\theta} = \int \frac{d\theta}{\tan\theta\sec^2\theta}$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$= \int \frac{\cos^3\theta}{\sin\theta}\,d\theta = \int \frac{(1-\sin^2\theta)\cos\theta}{\sin\theta}\,d\theta$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$= \int \frac{1-u^2\,du}{u} = \int \frac{1}{u} - u\,du = \ln|u| - \frac{1}{2}u^2 + C$

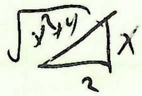
$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \boxed{\ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\frac{x^2}{x^2+1} + C}$

(4)

(j) $\displaystyle\int \frac{x^3}{(x^2+4)^{1/2}}\,dx = \int \frac{16\tan^3\theta\sec^2\theta\,d\theta}{2\sec\theta} = 8\int \tan^3\theta\sec\theta\,d\theta$

$x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$

$= 8\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta = \frac{8}{3}\sec^3\theta - 8\sec\theta + C$



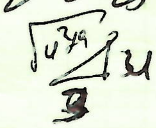
$= \frac{8}{3}\frac{(x^2+4)^{3/2}}{8} - 8\frac{\sqrt{x^2+4}}{2} + C = \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C}$

(k) $\displaystyle\int \frac{2}{\sqrt{25-4x^2}}\,dx = \int \frac{2\,dx}{\sqrt{4(\frac{25}{4}-x^2)}} = \int \frac{dx}{(\frac{25}{4}-x^2)^{1/2}} = \arcsin(2x/5) + C$

(l) $\displaystyle\int \frac{e^x}{(e^{2x}+9)^{1/2}}\,dx = \int \frac{du}{(u^2+9)^{1/2}} = \int \frac{3\sec^2\theta\,d\theta}{3\sec\theta} = \int \sec\theta\,d\theta$

$u = e^x$
$du = e^x\,dx$

$u = 3\tan\theta$
$du = 3\sec^2\theta\,d\theta$



$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| + C$

$= \ln\left|\frac{(e^{2x}+9)^{1/2}}{3} + \frac{e^x}{3}\right| + C$

$= \boxed{\ln|(e^{2x}+9)^{1/2} + e^x| - \ln 3 + C}$ constant

(m) $\displaystyle\int \sin^3 x\cos^2 x\,dx = \int(1-\cos^2 x)\cos^2 x\sin x\,dx = \boxed{-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$

(n) $\displaystyle\int \cos(2x)\sin^9(2x)\,dx = \frac{1}{10}\cdot\frac{1}{2}\sin^{10}(2x) + C$

(o) $\displaystyle\int 3xe^x\,dx = 3xe^x - 3\int e^x\,dx = \boxed{3xe^x - 3e^x + C}$

$u = 3x$ $\quad v = e^x$
$du = 3\,dx$ $\quad dv = e^x\,dx$