MATH 132 Worksheet 4 L'Hopital's Rule and Improper Integrals

1. L'Hopital's Rule

   (a) Basic Problems

      i. $\displaystyle\lim_{x\to 2}\frac{x-2}{x^2-4}$

      ii. $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x}$

   (b) One Basic Trick

      i. $\displaystyle\lim_{x\to 0}\frac{8x^2}{\cos(x)-1}$

      ii. $\displaystyle\lim_{x\to\infty}\frac{\ln(x)}{2\sqrt{x}}$

      iii. $\displaystyle\lim_{x\to\infty}\frac{e^{3x}}{x^3}$

   (c) More tricky - Advanced

      i. $\displaystyle\lim_{x\to 0}\frac{x\sin(x)}{1-\cos(x)}$

      ii. $\displaystyle\lim_{x\to 0^+}\sqrt{x}\ln(x)$

      iii. $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin(x)}-\frac{1}{x}\right)$

      iv. $\displaystyle\lim_{x\to 0^+}(1+x)^{1/x}$

      v. $\displaystyle\lim_{x\to\infty}(x-\sqrt{x^2+12})$

      vi. $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

2. Improper Integrals: Evaluate the following integral. Note: some of these may diverge - justify

   (a) $\displaystyle\int_0^\infty e^{-x}\,dx$

   (b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

   (c) $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$

   (d) $\displaystyle\int_4^\infty \frac{1}{x}\,dx$

   (e) $\displaystyle\int_0^\infty xe^{-x}\,dx$

   (f) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

   (g) $\displaystyle\int_0^1 \frac{1}{x^{3/2}}\,dx$

   (h) $\displaystyle\int_0^4 \frac{1}{x}\,dx$

   (i) $\displaystyle\int_0^\infty \frac{1}{x^2+3x+2}\,dx$

① ⓐ i $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$. or $\overset{LHR}{=} \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}$.

ⓐ ii) $\lim_{x\to 0} \frac{\sin(5x)}{x} \overset{LHR}{=} \lim_{x\to 0} \frac{5\cos x}{1} = 5$

ⓑ i) $\lim_{x\to 0} \frac{8x^2}{\cos(x)-1} \overset{LHR}{=} \lim_{x\to 0} \frac{16x}{-\sin x} \overset{LHR}{=} \lim_{x\to 0} \frac{16}{-\cos x} = -16$

ii) $\lim_{x\to \infty} \frac{\ln x}{2\sqrt{x}} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

iii) $\lim_{x\to\infty} \frac{e^{3x}}{x^3} \overset{LHR}{=} \lim_{x\to\infty} \frac{3e^{3x}}{3x^2} \overset{LHR}{=} \lim_{x\to\infty} \frac{9e^{3x}}{6x} = \lim_{x\to\infty} \frac{27e^{3x}}{6} = \infty$.

ⓒ i) $\lim_{x\to 0} \frac{x\sin x}{1-\cos x} \overset{LHR}{=} \lim_{x\to 0} \frac{x\cos x + \sin x}{\sin x} = \lim_{x\to 0} \frac{x\cos x}{\sin x} + 1$

$\overset{LHR}{=} \lim_{x\to 0} \frac{-x\sin x + \cos x}{\cos x} + 1 = \lim_{x\to 0} \frac{x\sin x}{\cos x} + 1 + 1 = 2$

or $= \lim_{x\to 0} \frac{x}{\sin x} \cdot \cos x + 1 = 1 \cdot 1 + 1 = 2$

ii) $\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2\frac{x^{3/2}}{x} = \lim_{x\to 0^+} -2\sqrt{x} = 0$

iii) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \overset{LHR}{=} \lim_{x\to 0} \frac{1-\cos x}{x\cos x - \sin x} \overset{LHR}{=} \lim_{x\to 0} \frac{+\sin x}{-x\sin x + \cos x + \cos x} = \frac{0}{2} = 0$

iv) $\lim_{x\to 0^+} (1+x)^{1/x}$: $\lim_{x\to 0^+} \ln(1+x)^{1/x} = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1$

so $\lim_{x\to 0^+} (1+x)^{1/x} = e^1 = e$.

v) $\lim_{x\to\infty} \frac{(x-\sqrt{x^2+12})(x+\sqrt{x^2+12})}{(x+\sqrt{x^2+12})} = \lim_{x\to\infty} \frac{x^2-(x^2+12)}{x+\sqrt{x^2+12}} = \lim_{x\to\infty} \frac{-12}{x+\sqrt{x^2+12}} = 0$

vi) $\lim_{x\to\infty} \ln(1+\frac{2}{x})^x = \lim_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}} \cdot \frac{-2}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$

so $\lim_{x\to\infty} (1+\frac{2}{x})^x = e^2$.

(2)

(a) $\int_0^\infty e^{-x}dx = \lim_{a\to\infty} \int_0^a e^{-x}dx = \lim_{a\to\infty} -e^{-x}\Big|_0^a = \lim_{a\to\infty} -e^{-a}+e^0 = 0+1=1$

(b) $= \lim_{a\to0^+} \int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to0^+} +2\sqrt{x}\Big|_a^1 = \lim_{a\to0^+} +2\sqrt{1} - 2\sqrt{a} = 2$

(c) $= \lim_{a\to\infty} \int_1^a \frac{1}{x^{3/2}}dx = \lim_{a\to\infty} -2x^{-1/2}\Big|_1^a = \lim_{a\to\infty} -2a^{-1/2}+2\cdot1 = 2$

(d) $= \lim_{a\to\infty} \int_4^a \frac{1}{x}dx = \lim_{a\to\infty} \ln|x|\Big|_4^a = \lim_{a\to\infty} \ln a - \ln 4 = \infty$, diverges,

(e) $= \lim_{a\to\infty} \int_0^a xe^{-x}dx = \dots = \lim -xe^{-x} - e^{-x}\Big|_0^a = \lim_{a\to\infty} -ae^{-a} - e^{-a} - (0 - e^0) \overset{\text{L'H}}{=} \lim_{a\to\infty} -\frac{1}{-e^a}^{0} + 1 = 1$

$u = x \quad v = -e^{-x}$

$du = dx \quad dv = e^{-x}dx$

(f) $= \lim_{a\to\infty} \int_1^a \frac{1}{\sqrt{x}}dx = \lim 2\sqrt{x}\Big|_1^a = \lim_{a\to\infty} 2\sqrt{a} - 2 = \infty$, diverges

(g) $= \lim_{a\to0^+} \int_a^1 x^{-3/2}dx = \lim_{a\to0^+} -2x^{-1/2}\Big|_a^1 = \lim_{a\to0^+} -2\cdot1 + 2/\sqrt{a} = \infty$, diverges

(h) $= \lim_{a\to0^+} \int_a^4 \frac{1}{x}dx = \lim_{a\to0^+} \ln 4 - \ln a = \infty$, diverges

(i) $= \lim_{a\to\infty} \int_0^a \frac{1}{(x+1)(x+2)}dx = \lim_{a\to\infty} \frac{1}{x+1} - \frac{1}{x+2}\,dx$ ← I checked here – did it w/out partial fraction decomp.

$= \lim_{a\to\infty} \ln|x+1| - \ln|x+2|\Big|_0^a = \lim_{a\to\infty} \ln\left(\frac{x+1}{x+2}\right)\Big|_0^a = \cancel{\lim_{a\to\infty} \frac{a+1}{a+2} -}$

$= \lim_{a\to\infty} \ln\left(\frac{a+1}{a+2}\right) - \ln\left(\frac{1}{2}\right) = \ln 1 + \ln 2 = \ln 2.$