MATH 132 Worksheet 5 Integration and Limits Review

1. Determine whether the following integrals converge or diverge.

(a) $\int_1^\infty e^{-x^3}\,dx$

(b) $\int_3^\infty \frac{1}{x^{1.1}}\,dx$

(c) $\int_3^\infty \frac{1}{x^{0.9}}\,dx$

(d) $\int_3^\infty \frac{1}{\sqrt{x^2-1}}\,dx$

(e) $\int_3^\infty \frac{1}{\sqrt{x^2+1}}\,dx$

(f) $\int_5^\infty \frac{1}{(x+4)^{3/2}}\,dx$

(g) $\int_5^\infty \frac{1}{(x-4)^{3/2}}\,dx$

(h) $\int_4^\infty \frac{x}{x^3-1}\,dx$

(i) $\int_4^\infty \frac{x}{x^3+1}\,dx$

2. Evaluate the following integrals.

(a) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(b) $\int_0^1 \frac{1}{x^{3/2}}\,dx$

(c) $\int_0^4 \frac{1}{x}\,dx$

3. Evaluate the following limits

(a) $\lim_{x\to\infty} \frac{x^{10}}{e^x}$

(b) $\lim_{x\to\infty} \frac{(\ln(x))^2}{\sqrt{x}}$

(c) $\lim_{x\to\infty} \left(1+\frac{e}{x}\right)^x$

(d) $\lim_{x\to 0^+} \sqrt{x}\ln(x)$

(e) $\lim_{x\to 0} \frac{8x^2}{\cos(x)-1}$

(f) $\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}}$

(g) $\lim_{x\to\infty} \frac{e^{3x}}{x^3}$

(h) $\lim_{x\to\infty} (2x-\sqrt{4x^2+7x+12})$

4. Evaluate the following integral. Note: some of these may diverge - justify

(a) $\int \frac{x}{\sqrt{16-x^2}}\,dx$

(b) $\int \sin^3(x)\cos^4(x)\,dx$

(c) $\int \frac{1}{x(9-x^2)}\,dx$

(d) $\int \tan^4(x)\sec^2(x)\,dx$

(e) $\int_0^\infty xe^{-x}\,dx$

(f) $\int_0^\infty xe^{-x^2}\,dx$

(g) $\int x\sqrt{16-x^2}\,dx$

(h) $\int \sqrt{16-x^2}\,dx$

(i) $\int \frac{x+1}{x^2+1}\,dx$

(j) $\int x\sqrt{16-x}\,dx$

(k) $\int \frac{1}{(2x+3)^2+16}\,dx$

(l) $\int_7^\infty \frac{1}{x^2-3x-4}\,dx$

(m) $\int (x^{3/2}+x^{1/2}-\frac{4}{\sqrt{x}})\,dx$

(n) $\int (2x+1)^2\,dx$

(o) $\int \sqrt{2x+7}\,dx$

(p) $\int \frac{\sin(x)}{\cos^4(x)}\,dx$

(q) $\int \frac{x+1}{x^2+x+1}\,dx$

(r) $\int \frac{2x}{x^2+1}\,dx$

(s) $\int (x+1)^2\sqrt{x}\,dx$

(t) $\int x\sin(x)\,dx$

(u) $\int x\sin^2(x)\,dx$

(v) $\int \frac{1}{x(1+\ln(x))}\,dx$

(w) $\int \cos^3(x)\sin^2(x)\,dx$

(x) $\int \cos^2(x)\sin^2(x)\,dx$

(1)

(a) $e^x < e^{3x}$ is clear when $x \geq 1$. Then $\frac{1}{e^{3x}} < \frac{1}{e^x}$ (dividing each side by e^x and e^{3x})

Notice that $\int_1^\infty \frac{1}{e^x}dx = \lim_{a\to\infty} \int_1^a e^{-x}dx = \lim_{a\to\infty} -(e^{-a} - e) = e$, so it converges. by the DCT, $\int_1^\infty \frac{1}{e^{-x3}}dx$ converges.

(b) $\int_3^\infty \frac{1}{x^{1.1}}dx < \int_1^\infty \frac{1}{x^{1.1}}dx$ which converges since $\overset{1.1}{\overset{\shortparallel}{p}} > 1$.

(c) $\int_3^\infty \frac{1}{x^{0.9}}dx = \int_1^\infty \frac{1}{x^{0.9}}dx - \int_1^3 \frac{1}{x^{0.9}}dx$ which diverges since the first integral diverges at the second is finite. (The second diverges because $p = 0.9 < 1$.

(d) Note that $\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2-1}}} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x} \cancel{\overset{LH}{=} \lim_{x\to\infty} \frac{2x}{2\sqrt{x^2-1}\cdot 1}}$ this isn't working.

$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2}} = \lim_{x\to\infty} \sqrt{\frac{x^2-1}{x^2}} = \lim_{x\to\infty} \sqrt{1-\frac{1}{x^2}} = 1.$

so, by LCT, $\int_3^\infty \frac{1}{\sqrt{x^2-1}}dx$ diverges since $\int_3^\infty \frac{1}{x}dx$ diverges (since $p=1$)

(e) almost the same as d. you can work it out.

(f) Notice that $\lim_{x\to\infty} \frac{\frac{1}{x^{3/2}}}{\frac{1}{(x+4)^{3/2}}} = \lim_{x\to\infty} \frac{(x+4)^{3/2}}{x^{3/2}} = \lim_{x\to\infty} \left(\frac{x+4}{x}\right)^{3/2}$

$= \lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{3/2} = 1^{3/2} = 1.$ Since $\int_5^\infty \frac{1}{x^{3/2}}dx$ converges ($p = 3/2 > 1$)

Then $\int_5^\infty \frac{1}{(x+4)^{3/2}}dx$ converges by LCT.

(g) almost the same as f. you can work it out.

(h) $\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{x}{x^3-1}} = \lim_{x\to\infty} \frac{x^3-1}{x^3} = \lim_{x\to\infty} 1 - \frac{1}{x^3} = 1$. Since $\int_4^\infty \frac{1}{x^2}dx$ converges ($p = 2 > 1$)

Then $\int_4^\infty \frac{x}{x^3-x}dx$ converges by LCT.

(i) again, similar to h.

(2) a) $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \int_a^1 x^{-1/2}dx = \lim_{a\to 0^+} 2x^{1/2}\Big|_a^1 = \lim_{a\to 0^+} 2\sqrt{1} - 2\sqrt{a} = 2.$

b) $\int_0^1 \frac{1}{x^{3/2}}dx = \lim_{a\to 0^+} \int_a^1 x^{-3/2}dx = \lim_{a\to 0^+} -2x^{-1/2}\Big|_a^1 = \lim_{a\to 0^+} \frac{-2}{\sqrt{1}} + \frac{2}{\sqrt{a}} = \infty$

so this one diverges.

c) $\int_0^4 \frac{1}{x}dx = \lim_{a\to 0^+} \int_a^4 \frac{1}{x}dx = \lim_{a\to 0^+} \ln|x|\Big|_a^4 = \lim_{a\to 0^+} \ln 4 - \ln a = \infty$

So this one diverges.

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(3) a) $\lim_{x\to\infty} \frac{x^{10}}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{10x^9}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{10\cdot 9\cdot x^8}{e^x} \overset{LHR}{=} \cdots \overset{LHR}{=} \lim_{x\to\infty} \frac{10!}{e^x} = 0.$

b) $\lim_{x\to\infty} \frac{(\ln(x))^2}{\sqrt{x}} \overset{LHR}{=} \lim_{x\to\infty} \frac{2\ln x\cdot \frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\ln x\cdot 2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{4\ln x}{\sqrt{x}}$

$\overset{LHR}{=} \lim_{x\to\infty} \frac{4\cdot\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{4\cdot 2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{8}{\sqrt{x}} = 0.$

c) First, notice that $\lim_{x\to\infty} \ln\left(1+\frac{e}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{e}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{e}{x}\right)}{\frac{1}{x}}$

$\overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{e}{x}}\cdot\left(\frac{-e}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{+e}{1+\frac{e}{x}} = e,$

Thus, $\lim_{x\to\infty} \left(1+\frac{e}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{e}{x}\right)^x} = e^e.$

d) $\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\overset{\to -\infty}{\ln x}}{\underset{\to\infty}{x^{-1/2}}} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} \frac{-2x^{3/2}}{x} = \lim_{x\to 0^+} -2\sqrt{x} = 0.$

e) $\lim_{x\to 0} \frac{\overset{\to 0}{8x^3}}{\underset{\to 0}{\cos x - 1}} \overset{LHR}{=} \lim_{x\to 0} \frac{16x}{-\sin x} = \lim_{x\to 0} -16\frac{x}{\sin x} = -16$ (recall $\lim_{x\to 0}\frac{x}{\sin x} = 1$)

f) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$

g) $\lim_{x\to\infty} \frac{e^{3x}}{x^3} \overset{LHR}{=} \lim_{x\to\infty} \frac{3e^{3x}}{3x^2} \overset{LHR}{=} \lim_{x\to\infty} \frac{9e^{3x}}{6x} \overset{LHR}{=} \lim_{x\to\infty} \frac{27e^{3x}}{6} = \infty.$

③ h) $\lim_{x\to\infty} 2x-\sqrt{4x^2+7x+12} = \lim_{x\to\infty} \frac{(2x-\sqrt{4x^2+7x+12})(2x+\sqrt{4x^2+7x+12})}{2x+\sqrt{4x^2+7x+12}}$

$= \lim_{x\to\infty} \frac{4x^2-(4x^2+7x+12)}{2x+\sqrt{4x^2+7x+12}} = \lim_{x\to\infty} \frac{-7x-12}{2x+\sqrt{4x^2+7x+12}}$

$= \lim_{x\to\infty} \frac{-7-\frac{12}{x}}{2+\frac{\sqrt{4x^2+7x+12}}{x}} = \lim_{x\to\infty} \frac{-7-\frac{12}{x}}{2+\sqrt{4+\frac{7}{x}+\frac{12}{x^2}}} = \frac{-7}{2+\sqrt{4}} = -\frac{7}{4}.$

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④ a) $\int \frac{x}{\sqrt{16-x^2}}dx = \int \frac{4\sin\theta \, 4\cos\theta \, d\theta}{(16-16\sin^2\theta)^{1/2}} = \int \frac{16\sin\theta\cos\theta \, d\theta}{[16(1-\sin^2\theta)]^{1/2}}$

$x = 4\sin\theta$
$dx = 4\cos\theta \, d\theta$

$= \int \frac{16\sin\theta\cos\theta \, d\theta}{4(\cos^2\theta)^{1/2}} = \int \frac{4\sin\theta\cos\theta \, d\theta}{\cos\theta} = \int 4\sin\theta \, d\theta = -4\cos\theta + c$

$= -\frac{4\sqrt{16-x^2}}{4} + c = \boxed{-\sqrt{16-x^2}+c}$

OR $u = 16-x^2$, $du = -2x\,dx$ $= \int -\frac{1}{2}u^{-1/2}du = -u^{1/2}+c = -(16-x^2)^{1/2}+c.$

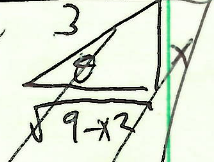
b) $\int \sin^3 x\cos^4 x \, dx = \int \cos^4 x(1-\cos^2 x)\sin x \, dx = \int (\cos^4 x - \cos^6 x)\sin x \, dx$

$\boxed{= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c}$ $\left[\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}\right]$

c) $\int \frac{1}{x(9-x^2)}dx = \int \frac{3\cos\theta \, d\theta}{3\sin\theta(9-9\sin^2\theta)} = \int \frac{3\cos\theta \, d\theta}{3\sin\theta \, 9\cos^2\theta} = \frac{1}{9}\int \frac{1}{\sin\theta}d\theta$

$x = 3\sin\theta$
$dx = 3\cos\theta \, d\theta$

$= \frac{1}{9}\int \csc\theta \, d\theta = -\frac{1}{9}\ln|\csc\theta + \cot\theta| + c$

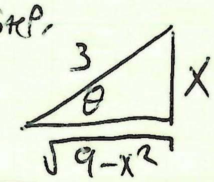


$= -\frac{1}{9}\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + c$

$= -\frac{1}{9}\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + c$

(4) c) Continued $= \int \frac{3\cos\theta \, d\theta}{3\sin\theta \, 9\cos^2\theta} = \int \frac{d\theta}{9\sin\theta\cos\theta} = \int \frac{1}{9}\frac{\sec^2\theta}{\tan\theta}\, d\theta$

↑ The tricky step.



$= \frac{1}{9}\ln|\tan\theta| + C = \frac{1}{9}\ln\left|\frac{x}{\sqrt{9-x^2}}\right| + C$

$= \frac{1}{9}\ln|x| - \frac{1}{9}\ln\left|\sqrt{9-x^2}\right| + C = \boxed{\frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C}$

OR Partial Fraction decomposition (answer looks different.)

$\dots = \int \frac{1}{9x} + \frac{1}{18(3-x)} + \frac{1}{18(3+x)}\, dx = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| + \frac{1}{18}\ln|3+x| + C$

[I feel like I might have a sign error here somewhere]

d) $\int \tan^4 x \sec^2 x \, dx = \boxed{\frac{1}{5}\tan^5 x + C}$ Nice, an easy one.

e) $\int_0^\infty x e^{-x}\, dx$. first, $\int x e^{-x}\, dx = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x}$.

$u = x \quad v = -e^{-x}$
$du = dx \quad dv = e^{-x}dx$

So we get $= \lim_{a\to\infty}\int_0^a x e^{-x}\, dx = \lim_{a\to\infty}\left(-xe^{-x} - e^{-x}\right)\Big|_0^a = \lim_{a\to\infty}\left(-ae^{-a} - e^{-a}\right) - \left(0 - e^{-0}\right)$

$= \lim_{a\to\infty} \frac{-a}{e^{+a}} - \frac{1}{e^a} + 1 = \boxed{1}$ (use LHR on first, second is clearly 0.)

f) $= \lim_{a\to\infty}\int_0^a x e^{-x^2}\, dx = \lim_{a\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^a = \lim_{a\to\infty} -\frac{1}{2}\left(\frac{1}{e^{a^2}} - \frac{1}{e^{0^2}}\right)$

$= \lim_{a\to\infty} -\frac{1}{2}(0 - 1) = \boxed{\frac{1}{2}}$

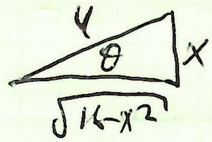
g) $\int x\sqrt{16-x^2}\, dx = \int -\frac{1}{2}u^{1/2}\, du = \boxed{-\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(16-x^2)^{3/2} + C}$

$u = 16 - x^2$
$du = -2x\, dx$
$\frac{du}{-2} = x\, dx$

OR trig sub, but this is easier.

(4) h) $\int \sqrt{16-x^2}\,dx = \int (16-16\sin^2\theta)^{1/2}\, 4\cos\theta\, d\theta = 16\int \cos^2\theta\, d\theta$

$x = 4\sin\theta$
$dx = 4\cos\theta\, d\theta$



$= 16\int \frac{1}{2}(1+\cos 2\theta)\, d\theta = 8\theta + 4\sin 2\theta + c$

$= 8\arcsin\left(\frac{x}{4}\right) + 8\sin\theta\cos\theta + c$

$= 8\arcsin\left(\frac{x}{4}\right) + 8\,\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + c$

$= \boxed{8\arcsin\left(\frac{x}{4}\right) + \frac{1}{2}x\sqrt{16-x^2} + c.}$

i) $\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \boxed{\frac{1}{2}\ln|x^2+1| + \arctan x + c}$

j) $\int x\sqrt{16-x}\,dx = -\int (16-u)u^{1/2}\,du = -\int 16u^{1/2} - u^{3/2}\,du$

$u = 16-x$
$du = -dx$
$x = 16-u$

$= -16\cdot\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c$

$= \boxed{-\frac{32}{3}(16-x)^{3/2} + \frac{2}{5}(16-x)^{5/2} + c}$

k) $\int \frac{1}{(2x+3)^2+16}\,dx = \frac{1}{2}\int \frac{1}{u^2+16}\,du = \frac{1}{2}\cdot\frac{1}{4}\arctan\left(\frac{u}{4}\right) + c$

$u = 2x+3$
$du = 2dx$

$= \boxed{\frac{1}{8}\arctan\left(\frac{2x+3}{4}\right) + c}$

l) Start with $\frac{1}{x^2-3x-4} = \frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$, so $1 = A(x+1) + B(x-4)$

or $1 = (A+B)x + (A-4B)$, so $A+B=0$ and $A-4B=1$

$B=-A$, $A-4(-A)=1$, $5A=1$, $\boxed{A=1/5,\ B=-1/5}$

so

$= \lim_{a\to\infty} \int_7^a \frac{1}{5}\frac{1}{x-4} - \frac{1}{5}\frac{1}{x+1}\,dx = \lim_{a\to\infty} \frac{1}{5}\ln|x-4|\Big|_7^a - \frac{1}{5}\ln|x+1|\Big|_7^a$

$= \lim_{a\to\infty} \frac{1}{5}(\ln|a-4| - \ln 3) - \left[\frac{1}{5}(\ln|a+1| - \ln 8)\right] = \lim_{a\to\infty} \frac{1}{5}\left[\ln 8 - \ln 3 + \ln|a-4| - \ln|a+1|\right]$

$= \lim_{a\to\infty} \frac{1}{5}\left[\ln(8/3) + \ln\left|\frac{a-4}{a+1}\right|\right] = \frac{\ln(8/3)}{5}$ $\left[\frac{a-4}{a+1} \to 1, \text{ so } \ln\frac{a-4}{a+1} \to 0\right]$

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m) $= \boxed{\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - 8x^{1/2} + c}$

n) $= \int 4x^2 + 4x + 1\,dx = \boxed{\frac{4}{3}x^3 + 2x^2 + x + c}$

o) $= \boxed{\frac{1}{3}(2x+7)^{3/2} + C}$

p) $\int \frac{\sin(x)}{\cos^4(x)}\,dx = \int \cos^{-4}x \sin x\,dx = \boxed{\frac{1}{3}\cos^{-3}x + c}$

q) $\int \frac{x+1}{x^2+x+1}\,dx = \int \frac{x+\frac{1}{2}}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$

$u = x^2+x+1$
$du = (2x+1)\,dx$

$= \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx$

$= \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}} \arctan\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + c$

$= \boxed{\frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\left((2x+1)/\sqrt{3}\right) + C}$

r) $= \boxed{\ln|x^2+1| + c}$

s) $\int (x+1)^2\sqrt{x}\,dx = \int (x^2+2x+1)\sqrt{x}\,dx = \int x^{5/2} + 2x^{3/2} + x^{1/2}\,dx$

$= \boxed{\frac{2}{7}x^{7/2} + \frac{4}{5}x^{5/2} + \frac{2}{3}x^{3/2} + c}$

t) $\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = \boxed{-x\cos x + \sin x + c}$

$u = x \quad v = -\cos x$
$du = dx \quad dv = \sin x\,dx$

u) $\int x \sin^2 x\,dx = \int x\,\frac{1}{2}(1 - \cos 2x)\,dx = \frac{1}{2}\int x - x\cos 2x\,dx$

$u = x \quad v = \frac{1}{2}\sin 2x$
$du = dx \quad dv = \cos 2x\,dx$

$= \frac{1}{2}\left(\frac{1}{2}x^2 - \left(\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right)\right)$

$= \boxed{\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c}$

v) $\int \frac{1}{x(1+\ln x)}\,dx = \int \frac{1}{u}\,du = \ln|u| = \boxed{\ln|1+\ln x| + c}$

$u = 1 + \ln x$
$du = \frac{1}{x}\,dx$

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(4) w) $\int \cos^3(x)\sin^2(x)\,dx = \int \sin^2 x\,(1-\sin^2 x)\cos x\,dx = \int(\sin^2 x - \sin^4 x)\cos x\,dx$

$= \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$

x) $\int \cos^2 x \sin^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{1}{4}\int (1-\cos^2 2x)\,dx = \frac{1}{4}\int 1-\frac{1}{2}(1+\cos 4x)\,dx = \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos 4x\,dx$

$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right) + C = \boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x + C}$