1. Write out the first 5 terms of the sequence (beginning with $n = 1$). Do you think it will converge? To what?

(a) $a_n = \left\{\frac{1}{n}\right\}$ (c) $a_n = \{3^n\}$ (e) $x_n = n^{\frac{1}{n}}$ (g) $e_n = \left(1 + \frac{2}{n}\right)^n$

(b) $a_n = \left\{\left(\frac{1}{2}\right)^n\right\}$ (d) $a_n = \left(1 - \frac{1}{n}\right)^n$ (f) $y_n = \left(\frac{2}{5}\right)^n$

2. Do the following sequences converge? Find the limit of each convergent sequence.

(a) $b_n = \left\{\left(\frac{1}{4}\right)^n\right\}$ (d) $e_n = \left(1 + \frac{2}{n}\right)^n$ (g) $b_n = \sqrt{\frac{2n}{n+1}}$

(b) $a_n = \frac{\ln(n)}{n}$ (e) $p_n = \left\{\frac{2n^2 - n + 3}{n^2 + 4n + 1}\right\}$ (h) $g_n = \{(-1)^n\}$

(c) $x_n = \frac{(\ln(n^2+1))^2}{n}$ (f) $a_n = \frac{n + (-1)^n}{n}$ (i) $y_n = \left(1 + \frac{7}{n}\right)^n$

3. Let $S_n = \sum_{k=1}^{n} a_k$. For each of the following, find a_1, a_2, a_3, a_4 and S_1, S_2, S_3, S_4.

(a) $a_n = \left(\frac{1}{2}\right)^n$ (b) $a_n = \frac{1}{n}$ (c) $a_n = \frac{1}{n^2}$ (d) $a_n = \left(\frac{2}{3}\right)^n$

4. Do the following series converge? Find the sum of each convergent series.

(a) $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$ (d) $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$ (g) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

(b) $\sum_{n=6}^{\infty} \left(\frac{e}{\pi}\right)^n$ (e) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$ (h) $\sum_{n=1}^{\infty} e^{2n}$

(c) $\sum_{n=0}^{\infty} 7\left(\frac{8}{9}\right)^n$ (f) $\sum_{n=1}^{\infty} (\sqrt{3})^n$ (i) $\sum_{n=1}^{\infty} e^{-2n}$

① a) $a_1=\frac{1}{1}=1$, $a_2=\frac{1}{2}$, $a_3=\frac{1}{3}$, $a_4=\frac{1}{4}$, $a_5=\frac{1}{5}$. Converges to 0.

b) $a_1=\frac{1}{2}$, $a_2=\frac{1}{4}$, $a_3=\frac{1}{8}$, $a_4=\frac{1}{16}$, $a_5=\frac{1}{32}$. Converges to 0

c) $a_1=3$, $a_2=9$, $a_3=27$, $a_4=81$, $a_5=243$. diverges

d) $a_1=(1-\frac{1}{1})^1=0$, $a_2=(1-\frac{1}{2})^2=(\frac{1}{2})^2=\frac{1}{4}$, $a_3=(1-\frac{1}{3})^3=(\frac{2}{3})^3=\frac{8}{27}$

$a_4=(1-\frac{1}{4})^4=(\frac{3}{4})^4=\frac{81}{256}$, $a_5=(1-\frac{1}{5})^5=(\frac{4}{5})^5=\frac{1024}{3125}$.

unclear pattern, but it can be realized with continuous function. Then

$$\lim_{n\to\infty} \ln(1-\tfrac{1}{n})^n = \lim_{n\to\infty} n\ln(1-\tfrac{1}{n}) = \lim_{n\to\infty} \frac{\ln(1-\frac{1}{n})}{\frac{1}{n}} \overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{n}}\cdot\left(\frac{1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} -\frac{1}{1-\frac{1}{n}} = -1. \text{ Thus, } \lim_{n\to\infty} (1-\tfrac{1}{n})^n = e^{-1} = \tfrac{1}{e}.$$

e) $x_1=1^1=1$, $x_2=2^{1/2}=\sqrt{2}$, $x_3=3^{1/3}$, $x_4=4^{1/4}$, $x_5=5^{1/5}$.

Limit is unclear. Consider the continuous function $x^{1/x}$ when $x>0$, then

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0, \text{ so}$$

$$\lim_{x\to\infty} x^{1/x} = e^0 = 1, \text{ so } \lim_{n\to\infty} x_n = 1.$$

f) $y_1=\frac{2}{5}$, $y_2=\frac{4}{25}$, $y_3=\frac{8}{125}$, $y_4=\frac{16}{625}$, $y_5=\frac{32}{3125}$. Converges to 0.

g) $e_1=(1+\frac{2}{1})^1=3$, $e_2=(1+\frac{2}{2})^2=2^2=4$, $e_3=(1+\frac{2}{3})^3=(\frac{5}{3})^3\approx 4.296$

$e_4=(1+\frac{2}{4})^4=(\frac{3}{2})^4\approx 5.0625$, $e_5=(1+\frac{2}{5})^5=(\frac{7}{5})^5\approx 5.37824$. unclear limit.

as before, we consider $$\lim_{x\to\infty} \ln(1+\tfrac{2}{x})^x = \lim_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} +\left(\frac{2}{1+\frac{2}{x}}\right) = 2, \text{ so } \lim_{x\to\infty} (1+\tfrac{2}{x})^x = e^2, \text{ so } e_n \text{ converges to } e^2$$

(2)

a) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} (\frac{1}{4})^n = \boxed{0}$

b) $\lim_{n\to\infty} \frac{\ln n}{n} \overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \boxed{0}.$

c) $\lim_{n\to\infty} \frac{[\ln(n^2+1)]^2}{n} \overset{LHR}{=} \lim_{n\to\infty} \frac{2\ln(n^2+1)\cdot\frac{1}{n^2+1}\cdot 2n}{1} = \lim_{n\to\infty} \frac{4\ln(n^2+1)\,n}{n^2+1}$

$\overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{4}{n^2+1}\cdot 2n\cdot n + 4\ln(n^2+1)}{2n} = \lim_{n\to\infty} \frac{4n}{n^2+1} + \frac{2\ln(n^2+1)}{n}$

$\overset{LHR}{=} \lim_{n\to\infty} \frac{4}{2n} + \frac{2\frac{1}{n^2+1}\cdot 2n}{1} = \lim_{n\to\infty} \frac{2}{n} + \frac{4n}{n^2+1} \overset{LHR}{=} 0 + \lim_{n\to\infty} \frac{4}{2n} = \boxed{0}.$

d) $e^2 \to$ see solution to 1g.

e) $p_n = \frac{2n^2-n+3}{n^2+4n+1} = \frac{2-\frac{1}{n}+\frac{3}{n^2}}{1+\frac{4}{n}+\frac{1}{n^2}}$, so $\lim_{n\to\infty} p_n = \frac{2}{1} = \boxed{2}.$

f) as n gets large, the ± 1 is insignificant. Clearly the limit is 1.

g) $\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{2n+2-2}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{2(n+1)}{n+1} - \frac{2}{n+1}}$

$= \lim_{n\to\infty} \sqrt{2-\frac{2}{n+1}} = \boxed{\sqrt{2}}.$

h) alternates between 1 and -1, so it does not converge.

i) $\lim_{n\to\infty} (1+\frac{7}{n})^n = 1^\infty$. indeterminate.

$\lim_{n\to\infty} \ln((1+\frac{7}{n})^n) = \lim_{n\to\infty} \overset{\nearrow\infty}{n}\,\ln(\overset{\nearrow 0}{1+\frac{7}{n}}) = \lim_{n\to\infty} \frac{\overset{\nearrow 0}{\ln(1+\frac{7}{n})}}{\underset{\searrow 0}{\frac{1}{n}}} \overset{LHR}{=}$

$\overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{7}{n}}\left(\frac{-7}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{7}{1+\frac{7}{n}} = 7.$

Thus $\lim_{n\to\infty} (1+\frac{7}{n})^n = \boxed{e^7}$

(3) a) $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$, $a_4 = \frac{1}{16}$. $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16}$

b) $a_n = \frac{1}{n}$ $a_1 = \frac{1}{1} = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{4}$

$S_1 = 1$, $S_2 = 1+\frac{1}{2} = \frac{3}{2}$, $S_3 = 1+\frac{1}{2}+\frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}$, $S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$

c) $a_n = \frac{1}{n^2}$ $a_1 = \frac{1}{1^2} = 1$, $a_2 = \frac{1}{2^2} = \frac{1}{4}$, $a_3 = \frac{1}{3^2} = \frac{1}{9}$, $a_4 = \frac{1}{4^2} = \frac{1}{16}$

$S_1 = 1$, $S_2 = 1+\frac{1}{4} = \frac{5}{4}$, $S_3 = 1+\frac{1}{4}+\frac{1}{9} = \frac{36+9+4}{36} = \frac{49}{36}$, $S_4 = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16} = \frac{144+36+16+9}{144} = \frac{205}{144}$

d) $a_n = \left(\frac{2}{3}\right)^n$, $a_1 = \frac{2}{3}$, $a_2 = \frac{4}{9}$, $a_3 = \frac{8}{27}$, $a_4 = \frac{16}{81}$. $S_1 = \frac{2}{3}$, $S_2 = \frac{2}{3}+\frac{4}{9} = \frac{6+4}{9} = \frac{10}{9}$

$S_3 = \frac{2}{3}+\frac{4}{9}+\frac{8}{27} = \frac{2\cdot 9+4\cdot 3+8}{27} = \frac{38}{27}$, $S_4 = \frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81} = \frac{2\cdot 27+4\cdot 9+8\cdot 3+16}{81} = \frac{130}{81}$.

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(4) a) $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n - \left(\frac{1}{4}\right)^0 = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3/4} - 1 = \frac{4}{3} - 1 = \boxed{\frac{1}{3}}$ because $\frac{1}{4} < 1$

b) $\sum_{n=6}^{\infty}\left(\frac{e}{\pi}\right)^n = \sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n - \left(\frac{e}{\pi}\right)^0 - \left(\frac{e}{\pi}\right)^1 - \left(\frac{e}{\pi}\right)^2 - \left(\frac{e}{\pi}\right)^3 - \left(\frac{e}{\pi}\right)^4 - \left(\frac{e}{\pi}\right)^5$ because $\frac{e}{\pi} < 1$

$$= \boxed{\frac{1}{1-\frac{e}{\pi}} - 1 - \frac{e}{\pi} - \left(\frac{e}{\pi}\right)^2 - \left(\frac{e}{\pi}\right)^3 - \left(\frac{e}{\pi}\right)^4 - \left(\frac{e}{\pi}\right)^5}$$

c) $\sum_{n=0}^{\infty} 7\left(\frac{8}{9}\right)^n = 7\sum_{n=0}^{\infty}\left(\frac{8}{9}\right)^n = 7\left(\frac{1}{1-8/9}\right) = 7\cdot\frac{1}{1/9} = 7\cdot 9 = \boxed{63}$

d) $\sum_{n=0}^{\infty}\left(\frac{5}{2^n} - \frac{1}{3^n}\right) = 5\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 5\frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 5\cdot\frac{1}{\frac{1}{2}} - \frac{1}{\frac{2}{3}} = 10 - \frac{3}{2} = \boxed{\frac{17}{2}}$

e) Let $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$. $S_1 = 1 - \frac{1}{\sqrt{2}}$, $S_2 = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} = 1 - \frac{1}{\sqrt{3}}$, etc,

$S_n = 1 - \frac{1}{\sqrt{n+1}}$. Then $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{\sqrt{n+1}} = \boxed{1}$

(4) f) $\sum_{n=1}^{\infty}(\sqrt{3})^n$ diverges since it is geometric with $r=\sqrt{3}>1$

g) noting that $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e} \neq 0$ (see 1d for this limit)

so by the nth term test, the series diverges.

h) $\lim_{n\to\infty} e^{2n} = \infty$, since this is not 0, the series diverges by nth term test.

i) $$\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} \frac{1}{e^{2n}} - e^{-2\cdot 0} = \sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^n - 1 = \frac{1}{1-\frac{1}{e^2}} - 1$$

$$= \frac{1}{\frac{e^2-1}{e^2}} - 1 = \frac{e^2}{e^2-1} - 1 = \frac{e^2-(e^2-1)}{e^2-1} = \boxed{\frac{1}{e^2-1}}.$$