MATH 132 Worksheet 7 nth-term, Integral, Direct Comparison, Limit Comparison Tests

1. Evaluate the following integrals and sketch the graphs of the functions/areas:

(a) $\int_1^{\infty} e^{-x}\,dx$ (b) $\int_1^{\infty} \frac{1}{x^2}\,dx$ (c) $\int_1^{\infty} \frac{1}{x}\,dx$

2. Add rectangles (above or below the curve as appropriate) to the sketches above that represent the sum of each series. Use this to determine the convergence of each series.

(a) $\sum_{n=1}^{\infty} e^{-n}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n}$

3. Use the n**th-term test** to show that the following series diverge.

(a) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{\sqrt{n}}$ (e) $\sum_{n=1}^{\infty} \left(\frac{15}{4}\right)^n$

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ (d) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

4. Determine the convergence/divergence of the following using the **integral test**.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ (c) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (e) $\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (f) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

5. Determine the convergence/divergence of the following using **direct comparison test**.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}-1/2}$ (e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-1/2}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$ (d) $\sum_{n=1}^{\infty} \frac{n}{n^2-1/2}$ (f) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+1}}$

6. Determine the convergence/divergence of the following using the **limit comparison test**.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2-1/2}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}+1/2}$ (e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1/2}}$

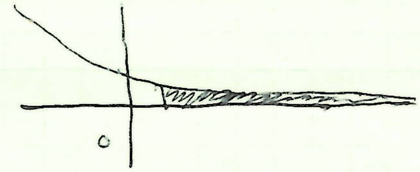
(b) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-1/2}$ (d) $\sum_{n=1}^{\infty} \frac{n}{n^2+1/2}$ (f) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5-1/2}}$

7. Determine the convergence/divergence of the following. If it converges, FIND ITS SUM!

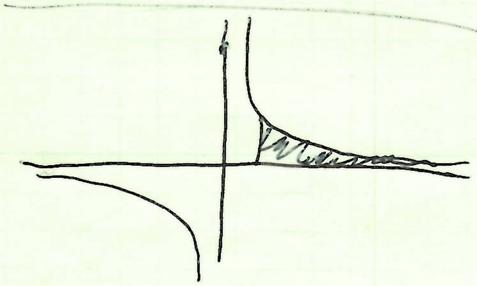
(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (c) $\sum_{n=3}^{\infty} \frac{2^n}{3^{2n}}$ (e) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

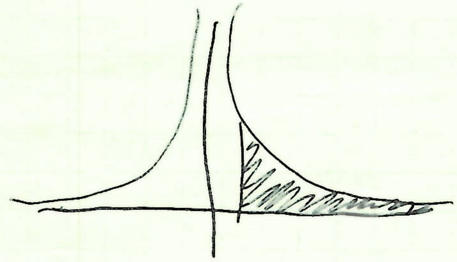
(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (d) $\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$ (f) $\sum_{n=1}^{\infty} 3^{-n}2^{n+2}$

① a) $\int_1^\infty e^{-x}\,dx = \lim_{a\to\infty}\int_1^a e^{-x}\,dx = \lim_{a\to\infty} -e^{-x}\Big|_1^a = \lim_{a\to\infty} -e^{-a}+e^{-1} = \boxed{\frac{1}{e}}$



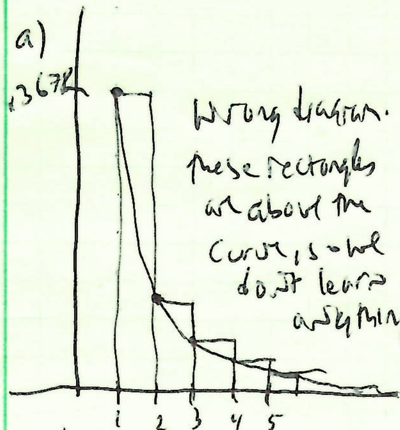
b) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty} -\frac{1}{x}\Big|_1^a = \lim_{a\to\infty} -\frac{1}{a}+\frac{1}{1} = 1$



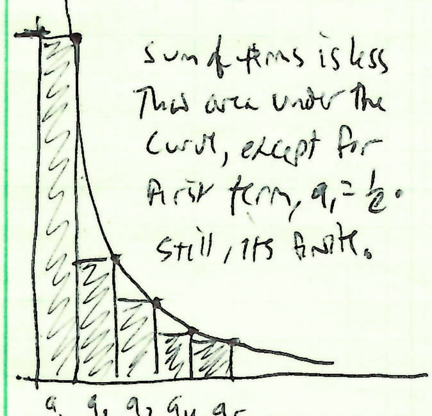


c) $\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln|a|\Big|_1^a = \lim_{a\to\infty} \ln a - \ln 1 = \infty.$

② a)



Wrong diagram. these rectangles are above the curve, so we don't learn anything



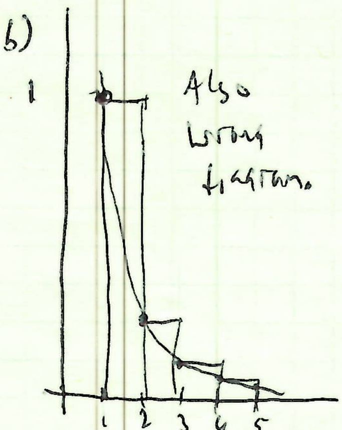
sum of terms is less than area under the curve, except for first term, $a_1 = \frac{1}{e}$. Still, it's finite.

$\sum_{n=1}^\infty e^{-n} = a_1 + \sum_{n=2}^\infty e^{-n} < \int_1^\infty e^{-x}\,dx = \frac{1}{e} + a_1$

(↑ $+a_n$)

so it converges

b)

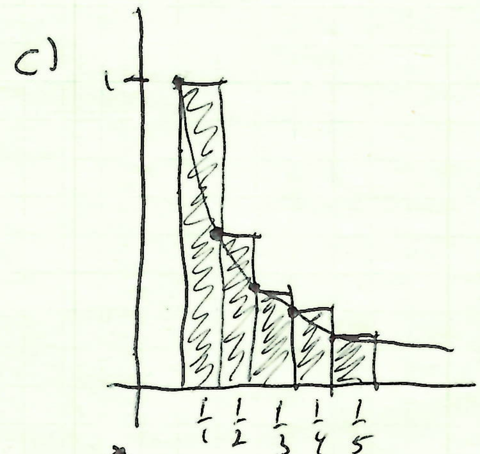


Also wrong diagram.

same idea as a).

This sum converges since the integral does and the area is less than, except for first term.

c)



↑ area of rectangles larger than area under the curve, so this summation diverges since the integral did

(3) a) $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = 1 \neq 0$, so by nth term test, $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ diverges.

b) $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{n^2 \cdot \frac{1}{n^2}}{(n^2+1)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \neq 0$, so by nth term test

$\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ diverges

c) $\lim_{n\to\infty} \frac{e^n}{\sqrt{n}} \overset{LHR}{=} \lim_{n\to\infty} \frac{e^n}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} 2e^n\sqrt{n} = \infty$, so by nth term test $\sum_{n=1}^{\infty} \frac{e^n}{\sqrt{n}}$ diverges.

d) We need $\lim_{n\to\infty} (1-\frac{1}{n})^n$. first, $\lim_{n\to\infty} \ln((1-\frac{1}{n})^n) = \lim_{n\to\infty} n \ln(1-\frac{1}{n})$

$= \lim_{n\to\infty} \frac{\ln(1-\frac{1}{n})}{\frac{1}{n}} \overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{n}} \cdot (\frac{1}{n^2})}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\frac{1}{1-\frac{1}{n}} = -1$

so $\lim_{n\to\infty} (1-\frac{1}{n})^n = e^{-1} \neq 0$, so by nth term test $\sum_{n=1}^{\infty} (1-1/n)^n$ diverges

e) $\lim_{n\to\infty} \left(\frac{15}{4}\right)^n = \infty$ since $\frac{15}{4} > 1$, so by nth term test $\sum_{n=1}^{\infty} \left(\frac{15}{4}\right)^n$ diverges.

④ a) $\int_1^\infty \frac{1}{x^{2/3}}dx$ diverges since $p = 2/3 < 1$, so by IT (integral test) $\sum_{n=1}^\infty \frac{1}{n^{2/3}}$ diverges

b) $\int_1^\infty \frac{1}{x^{3/2}}dx$ converges since $p > 1$, so by IT, $\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ converges

c) ~~$\int_1^\infty \frac{x}{x^2+1}dx = \lim_{a\to\infty} \int_1^a \frac{1}{2}\frac{1}{u}du$ $u = x^2+1$ $du = 2x\,dx$~~ let me be more careful...

$$\int_1^\infty \frac{x}{x^2+1}dx = \lim_{a\to\infty} \int_1^a \frac{x}{x^2+1}dx = \lim_{a\to\infty} \int_2^{a^2+1} \frac{1}{2}\frac{1}{u}du = \lim_{a\to\infty} \frac{1}{2}\ln|u|\Big|_2^{a^2+1}$$

$u = x^2+1$
$du = 2x\,dx$

$= \lim_{a\to\infty} \frac{1}{2}\ln(a^2+1) - \frac{1}{2}\ln 2 = \infty$, so the integral diverges. by IT, $\sum_{n=1}^\infty \frac{n}{n^2+1}$ diverges.

d) $$\int_1^\infty \frac{1}{x^2+1}dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2+1}dx = \lim_{a\to\infty} \arctan x\Big|_1^a = \lim_{a\to\infty} \arctan a - \arctan 1$$

$= \pi/2 - \pi/4 = \pi/4$, so $\sum_{n=1}^\infty \frac{1}{n^2+1}$ converges by IT.

e) $$\int_3^\infty \frac{1}{n\ln n}dn = \lim_{a\to\infty} \int_3^a \frac{1}{n\ln n}dn = \lim_{a\to\infty} \int_{\ln 3}^{\ln a} \frac{1}{u}du = \lim_{a\to\infty} \ln u\Big|_{\ln 3}^{\ln a}$$

$u = \ln n$
$du = \frac{1}{n}dn$

$= \lim_{a\to\infty} \ln\ln a - \ln\ln 3 = \infty$, so by IT, $\sum_{n=3}^\infty \frac{1}{n\ln n}$ diverges.

f) $$\int_1^\infty \frac{\ln x}{x}dx = \lim_{a\to\infty} \int_1^a \frac{\ln x}{x}dx = \lim_{a\to\infty} \int_0^{\ln a} u\,du = \lim_{a\to\infty} \frac{1}{2}u^2\Big|_0^{\ln a}$$

$u = \ln x$
$du = \frac{1}{x}dx$

$= \lim_{a\to\infty} \frac{1}{2}\ln^2 a - 0 = \infty$, so $\sum_{n=1}^\infty \frac{\ln n}{n}$ diverges by IT.

(5) a) $n^2+1 > n^2$, so $\frac{1}{n^2} > \frac{1}{n^2+1}$ (cross multiply the first inequality). direct comparison test

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by DCT.

b) notice that $n^{3/2}+1 > n^{3/2}$. Thus $\frac{1}{n^{3/2}} \geq \frac{1}{n^{3/2}+1}$ (or $\frac{1}{n^{3/2}+1} < \frac{1}{n^{3/2}}$ if you prefer)

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges because it is a p-series with $p=3/2>1$, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$ converges by DCT.

c) notice that $n^{2/3} > n^{2/3}-1/2$, so $\frac{1}{n^{2/3}} < \frac{1}{n^{2/3}-\frac{1}{2}}$ (if you can't see this logic by now, you should do some algebra to convince yourself that this is correct.)

$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a p-series with $p<1$, so it diverges

by DCT, $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}-\frac{1}{2}}$ also diverges (because its terms are all larger)

d) $n^2-\frac{1}{2} < n^2$, so $\frac{1}{n^2} < \frac{1}{n^2-\frac{1}{2}}$ multiplying both sides by n (we can assume $n>0$), $\frac{n}{n^2} < \frac{n}{n^2-\frac{1}{2}}$, or $\frac{n}{n^2-\frac{1}{2}} > \frac{1}{n}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$), $\sum_{n=1}^{\infty} \frac{n}{n^2-\frac{1}{2}}$ also diverges by DCT.

[again, we showed its terms were larger than those of a divergent sequence.]

e) if $n>0$, $n=\sqrt{n^2} > \sqrt{n^2-1/2}$, so $\frac{1}{\sqrt{n^2-\frac{1}{2}}} > \frac{1}{n}$. for same reasons as d), this diverges by DCT.

f) $\sqrt{n^5+1} > \sqrt{n^5} = n^{5/2}$, so $\frac{1}{\sqrt{n^5+1}} < \frac{1}{n^{5/2}}$. multiplying both sides by n,

$\frac{n}{\sqrt{n^5+1}} < \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p=3/2>1$), so by DCT (because our terms are smaller than those of a convergent sequence) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+1}}$ converges.

⑥ a) $\lim_{n\to\infty} \dfrac{\frac{1}{n^2}}{\frac{1}{n^2-\frac{1}{2}}} = \lim_{n\to\infty} \dfrac{n^2-\frac{1}{2}}{n^2} = \lim_{n\to\infty} 1 - \frac{1}{2n^2} = 1$, by LCT,

$\sum_{n=1}^{\infty} \frac{1}{n^2-\frac{1}{2}}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$) and the limit is finite and positive.

b) Similar to a, but compare with $\frac{1}{n^{3/2}}$. This one also converges

c)

$\lim_{n\to\infty} \dfrac{\frac{1}{n^{2/3}}}{\frac{1}{n^{2/3}+\frac{1}{2}}} = \lim_{n\to\infty} \dfrac{n^{2/3}+\frac{1}{2}}{n^{2/3}} = \lim_{n\to\infty} 1 + \frac{1}{2n^{2/3}} = 1.$

Since this is finite and positive, and since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (p-series with $p=\frac{2}{3}<1$),

$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}+\frac{1}{2}}$ diverges by LCT.

d) LCT with $\frac{1}{n}$. You can work out the details

e) Lets compare to $\frac{1}{n}$:

$\lim_{n\to\infty} \dfrac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+1/2}}} = \lim_{n\to\infty} \dfrac{\sqrt{n^2+1/2}}{n} = \lim_{n\to\infty} \dfrac{\sqrt{n^2+1/2}}{\sqrt{n^2}} = \lim_{n\to\infty} \sqrt{\dfrac{n^2+\frac{1}{2}}{n^2}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{2n^2}} = 1.$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$), $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1/2}}$ diverges by LCT.

f) n on top, $n^{5/2}$ on bottom (close to that anyway), so comparing with $1/n^{3/2}$ makes sense:

$\lim_{n\to\infty} \dfrac{\frac{1}{n^{3/2}}}{\frac{n}{\sqrt{n^5-1/2}}} = \lim_{n\to\infty} \dfrac{\sqrt{n^5-1/2}}{n^{3/2}} = \lim_{n\to\infty} \sqrt{\dfrac{n^5-\frac{1}{2}}{n^5}} = \lim_{n\to\infty} \sqrt{1-\frac{1}{2n^5}} = 1.$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p=3/2>1$), $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5-1/2}}$ converges by LCT.

(7) a) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \frac{1}{2^0} = \frac{1}{1-\frac{1}{2}} - 1 = 2-1 = \boxed{1}.$

b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{(n+1)}$ (you can check this by multiplying it out)

$S_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$. $S_1 = \frac{1}{1} - \frac{1}{2}$, $S_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$

aha! $S_n = 1 - \frac{1}{n+1}$. so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} 1 - \frac{1}{n+1} = \boxed{1}$

c) $\sum_{n=3}^{\infty} \frac{2^n}{3^{2n}} = \sum_{n=3}^{\infty} \left(\frac{2}{9}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^n - \left[\left(\frac{2}{9}\right)^0 + \left(\frac{2}{9}\right)^1 + \left(\frac{2}{9}\right)^2\right] = \frac{1}{1-\frac{2}{9}} - 1 - \frac{2}{9} - \frac{4}{81} = \frac{9}{7} - 1 - \frac{2}{9} - \frac{4}{81}$

$= (\text{do some algebra}) = \frac{8}{7\cdot 81} = \frac{8}{567}$

d) $\frac{3}{n^2+3n+2} = \frac{3}{n+1} - \frac{3}{n+2} = 3\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ (Partial fraction decomposition — details omitted.)

define $S_n = \sum_{k=1}^{n} \frac{3}{k^2+3k+2} = 3\sum_{k=1}^{n} \frac{1}{n+1} - \frac{1}{n+2}$.

times 3, not power.

$S_1 = \left(\frac{1}{2} - \frac{1}{3}\right)3$, $S_2 = \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}\right)3 = \left(\frac{1}{2} - \frac{1}{4}\right)3$, $S_3 = \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5}\right)3 = \left(\frac{1}{2} - \frac{1}{5}\right)3$

$S_n = \left(\frac{1}{2} - \frac{1}{n+2}\right)\cdot 3$ so $\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 3\left(\frac{1}{2} - \frac{1}{n+2}\right) = \boxed{3/2}$

e) $\frac{1}{n^2+4n+3} = \frac{1}{(n+1)(n+3)} = \left[\frac{1}{n+1} - \frac{1}{n+3}\right]\cdot\frac{1}{2}$ (Partial fraction decomposition — details omitted)

define $S_n = \sum_{k=1}^{n} \frac{1}{k+1} - \frac{1}{k+3}$. $S_1 = \frac{1}{2} - \frac{1}{4}$, $S_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$, $S_3 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6}$

$S_4 = S_3 + \frac{1}{5} - \frac{1}{7} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7}$, $S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$.

$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = \lim_{n\to\infty} \frac{1}{2} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{4} + \frac{1}{6} = \boxed{\frac{5}{12}}$

f) $\sum_{n=1}^{\infty} 3^{-n} 2^{n+2} = 4\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 4\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - 4 = 4\left(\frac{1}{1-\frac{2}{3}}\right) - 4 = 12 - 4 = \boxed{8}$