MATH 132 Worksheet 8 Choose your own series test

Determine convergence/divergence of each series. Justify your answers!

1. $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$

2. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

3. $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$

4. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$

5. $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^n$

6. $\sum_{n=1}^{\infty} \frac{2}{n^3}$

7. $\sum_{n=1}^{\infty} e^{-n}$

8. $\sum_{n=1}^{\infty} \frac{2n^2-n+3}{n^2+4n+1}$

9. $\sum_{n=1}^{\infty} \frac{n+(-1)^n}{n}$

10. $\sum_{n=1}^{\infty} (-1)^n$

11. $\sum_{n=1}^{\infty} \frac{5n^5+n^2-5n}{n^{15}-7n^{12}+8}$

12. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n+8}$

13. $\sum_{n=1}^{\infty} \frac{n^9-n^6+5\sqrt{n}}{n^{11}-n^5+6}$

14. $\sum_{n=1}^{\infty} \sqrt{\frac{2n}{n+1}}$

15. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

16. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

17. $\sum_{n=1}^{\infty} (\sqrt{3})^n$

18. $\sum_{n=1}^{\infty} e^{-2n}$

19. $\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$

20. $\sum_{n=4}^{\infty} \frac{1}{n^2-4n+5}$

21. $\sum_{n=1}^{\infty} \ln(n)$

22. $\sum_{n=1}^{\infty} \frac{n+2}{n^3+7n-1}$

23. $\sum_{n=1}^{\infty} \frac{3+n}{5+n}$

24. $\sum_{n=1}^{\infty} ne^{-n}$

25. $\sum_{n=1}^{\infty} \frac{n+2}{n^3+7n-1}$

26. $\sum_{n=2}^{\infty} \frac{n+5}{\sqrt{n^2-1}}$

27. $\sum_{n=1}^{\infty} \frac{3}{n(n+4)}$

28. $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

29. $\sum_{n=1}^{\infty} \frac{\sqrt{3n}}{(1+\sqrt{n})^5}$

30. $\sum_{n=1}^{\infty} \frac{n^{3/2}}{5n^2-1}$

31. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^8+3}}$

① $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n < \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$, so it converges. OR "its a geometric series with $|r|<1$."

② If $n \geq 3$, $\ln n > 1$, so $\frac{\ln n}{n} > \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (P-series, $p=1$),

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by DCT.

③ note that $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2n}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{2n^2} = \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2}$. Since this is positive and real and since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}\frac{2n}{n^2+1}$ diverges by LCT.

④ This is a p-series with $p<1$, so it diverges.

⑤ $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \cdots = e^2$ (use the log trick), so $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^n$ diverges by the n-th term test.

⑥ p-series with $p=3>1$, so it converges.

⑦ $= \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$, which converges since it is a geometric series with $|r|<1$.

⑧ $\lim_{n\to\infty} \frac{2n^2-n+3}{n^2+4n+1} = 2$ (L'hopital twice or algebra), so series diverges by nth term test.

⑨ $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1$, so series diverges by nth term test.

⑩ $\lim_{n\to\infty}(-1)^n$ does not exist, so it is not 0, so series diverges by nth term test.

⑪ $\lim_{n\to\infty} \frac{\frac{1}{n^{10}}}{\frac{5n^5+n^2-5n}{n^{15}-7n^{12}+8}} = \lim_{n\to\infty} \frac{n^{15}-7n^{12}+8}{5n^{15}+n^{12}-5n^{11}} = \lim_{n\to\infty} \frac{1-\frac{7}{n^3}+\frac{8}{n^{15}}}{5+\frac{1}{n^3}-\frac{5}{n^4}} = \frac{1}{5}$.

so by LCT $\sum_{n=1}^{\infty}\frac{5n^5+n^2-5n}{n^{15}-7n^{12}+8}$ converges since $\sum_{n=1}^{\infty}\frac{1}{n^{10}}$ converges (p series with $p=10>1$.)

⑫ $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{\sqrt{n}}{3n+8}} = \lim_{n\to\infty} \frac{3n+8}{n} = \lim_{n\to\infty} 3+\frac{8}{n} = 3$, so by LCT, $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{3n+8}$ diverges since

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (p-series with $p=\frac{1}{2}<1$).

(13) do LCT with $\frac{1}{n^2}$ to show it converges

(14) $\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{2}{1+\frac{1}{n}}} = \sqrt{2}$, so the series diverges by nth term test.

(15) $S_1 = \frac{1}{1} - \frac{1}{\sqrt{2}}$, $S_2 = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} = 1 - \frac{1}{\sqrt{3}}$, $S_3 = S_2 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} = 1 - \frac{1}{\sqrt{4}}$,

$S_n = 1 - \frac{1}{\sqrt{n+1}}$. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{\sqrt{n+1}} = 1$, so it converges.

(16) First, $\frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$, so $4 = 4An + A + 4Bn - 3B$,
$4A + 4B = 0$, $A - 3B = 4$, $B = -A$,
$A + 3A = 4$, $A = 1$, $B = -1$,

so we have

$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \frac{1}{4n-3} - \frac{1}{4n+1}.$$

$S_1 = 1 - \frac{1}{5}$, $S_2 = S_1 + \frac{1}{5} - \frac{1}{9} = 1 - \frac{1}{9}$, $S_3 = S_2 + \frac{1}{9} - \frac{1}{13} = 1 - \frac{1}{13}$, $S_n = 1 - \frac{1}{4n+1}$

So $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{4n+1} = 1$, so it converges.

(17) Geometric series with $|r| = \sqrt{3} > 1$, so it diverges

(18) $= \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n$, since $\frac{1}{e^2} < 1$, it converges because it is geometric series.

(19) $\int_3^{\infty} \frac{1}{x \ln x} dx = \lim_{a\to\infty} \int_3^a \frac{1}{x \ln x} dx = \lim_{a\to\infty} \int_{\ln 3}^{\ln a} \frac{1}{u} du = \lim_{a\to\infty} \ln|u| \Big|_{\ln 3}^{\ln a}$

$u = \ln x$
$du = \frac{1}{x} dx$

Note: $\frac{1}{x \ln x}$ is positive, decreasing, continuous on $[3, \infty)$

$= \lim_{a\to\infty} \ln \ln a - \ln \ln 3 = \infty$, so the series diverges by IT. (integral test)

(20) Use LCT with $\frac{1}{n^2}$ to show it converges.

(21) Use nth term test to show it diverges.

(22) Use LCT with $1/n^2$ to show it converges.

(23) Use nth term test to show it diverges.

(24) Use integral ~~test~~ - see next page.

(24) xe^{-x} is positive, decreasing, and continuous on $[1,\infty)$.

$$\int_1^\infty xe^{-x}\,dx = \lim_{a\to\infty}\int_1^a xe^{-x}\,dx = \lim_{a\to\infty} -xe^{-x}\Big|_1^a + \int_1^a e^{-x}\,dx$$

$u = x \quad v = -e^{-x}$

$du = dx \quad dv = e^{-x}dx$

$$= \lim_{a\to\infty} -xe^{-x} - e^{-x}\Big|_1^a = \lim_{a\to\infty} -ae^{-a} - e^{-a} + e^{-1} + e^{-1} = \lim_{a\to\infty}\frac{2}{e} - \frac{a+1}{e^a}$$

$$= \frac{2}{e} - \lim_{a\to\infty}\frac{a+1}{e^a} \stackrel{\text{LH}}{=} \frac{2}{e} - \lim_{a\to\infty}\frac{1}{e^a} = \frac{2}{e}.$$ By IT, the series converges.

(25) Oops - same as #22.

(26) $$\lim_{n\to\infty}\frac{n+5}{\sqrt{n^2-1}} = \lim_{n\to\infty}\sqrt{\frac{(n+5)^2}{n^2-1}} = \lim_{n\to\infty}\sqrt{\frac{n^2+10n+25}{n^2-1}} = \lim_{n\to\infty}\sqrt{\frac{1+\frac{10}{n}+\frac{25}{n^2}}{1-\frac{1}{n^2}}} = 1,$$

So by nth term test the series diverges.

(27) LCT with $1/n^2$ or partial fraction decomp to turn it into telescoping series.

Obviously, LCT is easier. Either way we can get that it converges.

(28) Use LCT with $1/\sqrt{n}$ to show it diverges

(29) Use LCT with $1/n^2$ to show it converges

(30) Use LCT with $1/\sqrt{n}$ to show it diverges

(31) $$\lim_{n\to\infty}\frac{\frac{1}{n^3}}{\frac{n}{\sqrt{n^8+3}}} = \lim_{n\to\infty}\frac{\sqrt{n^8+3}}{n^4} = \lim_{n\to\infty}\sqrt{\frac{n^8+3}{n^8}} = \lim_{n\to\infty}\sqrt{1+\frac{3}{n^8}} = 1,$$

So the series converges by LCT with $\frac{1}{n^3}$, where $\sum_{n=1}^\infty \frac{1}{n^3}$ converges because it is a p-series with $p=3>1$.