MATH 132 Worksheet 9 Series Tests: ratio (RaT), root (RoT), and others

1. Determine the convergence/divergence of the following series (mostly RaT):

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

(c) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

(d) $\sum_{n=1}^{\infty} \frac{5^n}{(n+1)4^{2n+1}}$

(e) $\sum_{n=1}^{\infty} ne^{-n}$

(f) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

(g) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

(h) $\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$

(i) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

(j) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(k) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

(l) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\dots(2n-1)}{n!}$

(m) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

(n) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$

2. Determine the convergence/divergence of the following series (lots of mixed problems):

(a) $\sum_{n=1}^{\infty} \frac{n^6}{6^n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)$

(c) $\sum_{n=1}^{\infty} \left(1-\frac{3}{n}\right)^n$

(d) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

(e) $\sum_{n=1}^{\infty} \frac{n!\ln(n)}{n(n+2)!}$

(f) $\sum_{n=2}^{\infty} \frac{n+5}{\sqrt{n^2-1}}$

(g) $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

(h) $\sum_{n=1}^{\infty} \frac{n^4}{n^6+3}$

(i) $\sum_{n=14}^{\infty} \frac{n6^n}{(n+1)!}$

(j) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

(k) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{\ln n^2}\right)^n$

(l) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(m) $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

(n) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

(o) $\sum_{n=1}^{\infty} \ln(n)$

(p) $\sum_{n=1}^{\infty} \frac{n+2}{n^3+7n-1}$

(q) $\sum_{n=1}^{\infty} \frac{16^n}{n^{120}}$

(r) $\sum_{n=1}^{\infty} \frac{3}{n(n+4)}$

(s) $\sum_{n=1}^{\infty} \frac{6+7^n}{9+7^n}$

(t) $\sum_{n=1}^{\infty} \frac{1}{6+\sqrt[4]{n^6}}$

(u) $\sum_{n=1}^{\infty} \frac{\sqrt{3n}}{(1+\sqrt{n})^5}$

(v) $\sum_{n=1}^{\infty} \frac{n^2+6n}{n^7+2}$

(w) $\sum_{n=1}^{\infty} \frac{n^{3/2}}{5n^2-1}$

(x) $\sum_{n=1}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$

(y) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

(z) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+2}}\right)$

3. Determine the convergence/divergence of the following series (challenging):

(a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

(b) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

(c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

(d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$

① a) $\lim_{n\to\infty} \dfrac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \dfrac{(n+1)^2}{n^2}\dfrac{2^n}{2^{n+1}} = \lim_{n\to\infty} \dfrac{n^2+2n+1}{2n^2} = \lim_{n\to\infty} \dfrac{1+\frac{2}{n}+\frac{1}{n^2}}{2} = \dfrac{1}{2} < 1,$

So the series converges by RaT (ratio test).

b) $\lim_{n\to\infty} \dfrac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} = \lim_{n\to\infty} \dfrac{(n+1)!\,10^n}{n!\,10^{n+1}} = \lim_{n\to\infty} \dfrac{(n+1)\,n!}{10\,n!} = \lim_{n\to\infty} \dfrac{n+1}{10} = \infty$, so series diverges by RaT.

c) $\lim_{n\to\infty} \dfrac{e^{n+1}}{(n+1)!} \Big/ \dfrac{e^n}{n!} = \lim_{n\to\infty} \dfrac{n!\,e^{n+1}}{(n+1)n!\,e^n} = \lim_{n\to\infty} \dfrac{e}{n+1} = 0 < 1$ so it converges by RaT.

d) $\lim_{n\to\infty} \dfrac{5^{n+1}}{(n+2)\,4^{2(n+1)+1}} \cdot \dfrac{(n+1)\,4^{2n+1}}{5^n} = \lim_{n\to\infty} \dfrac{n+1}{n+2}\,5 \cdot \dfrac{4^{2n+1}}{4^{2n+3}} = \lim_{n\to\infty} \dfrac{5}{16}\,\dfrac{n+1}{n+2} = \dfrac{5}{16} < 1$

so the series converges by RaT.

e) $\lim_{n\to\infty} \dfrac{(n+1)}{e^{n+1}} \cdot \dfrac{e^n}{n} = \lim_{n\to\infty} \dfrac{n+1}{en} = \lim_{n\to\infty} \dfrac{1+\frac{1}{n}}{e} = \dfrac{1}{e} < 1$, so series converges by RaT.

f) $\lim_{n\to\infty} \dfrac{(n+1)!}{(2(n+1))!} \cdot \dfrac{(2n)!}{n!} = \lim_{n\to\infty} \dfrac{(n+1)\,n!\,(2n)!}{n!\,(2n+2)!} = \lim_{n\to\infty} \dfrac{(n+1)(2n)!}{(2n+2)(2n+1)\,2n!} = \lim_{n\to\infty} \dfrac{n+1}{(2n+2)(2n+1)} = 0,$

So series converges by RaT.

g) RaT seems messy. Lets try nth term test for divergence.

$\lim_{n\to\infty} \ln((1-\tfrac{1}{n})^n) = \lim_{n\to\infty} \dfrac{\overset{\to 0}{\ln(1-\frac{1}{n})}}{\underset{\to 0}{\frac{1}{n}}} \overset{LHR}{=} \lim_{n\to\infty} \dfrac{\frac{1}{1-\frac{1}{n}} \cdot \left(\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\dfrac{1}{1-\frac{1}{n}} = -1$

so $\lim_{n\to\infty} (1-\tfrac{1}{n})^n = e^{-1} = \tfrac{1}{e} \neq 0$, so series diverges by nth term test.

h) $\lim_{n\to\infty} \dfrac{(n+1)^4\,4^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^4\,4^n} = \lim_{n\to\infty} \dfrac{4(n+1)^4}{(n+1)\,n^4} = \lim_{n\to\infty} \dfrac{4(n+1)^3}{n^4} = \cdots = 0 < 1$, so series converges by RaT.

Failed attempt at nth term test. See next page for more!

i) $\lim_{n\to\infty} \ln(1-\tfrac{1}{n})^{n^2} = \lim_{n\to\infty} n^2 \ln(1-\tfrac{1}{n}) = \lim_{n\to\infty} \dfrac{\overset{\to 0}{\ln(1-\frac{1}{n})}}{\underset{\to 0}{\frac{1}{n^2}}} \overset{LHR}{=} \lim_{n\to\infty} \dfrac{\frac{1}{1-\frac{1}{n}} \cdot \left(\frac{1}{n^2}\right)}{-\frac{2}{n^3}} = \lim_{n\to\infty} \dfrac{\frac{1}{1-\frac{1}{n}}}{-2/n}$

$= \lim_{n\to\infty} \dfrac{-n}{2(1-\frac{1}{n})} = \lim_{n\to\infty} \dfrac{-n}{2(\frac{n-1}{n})} = \lim_{n\to\infty} \dfrac{-n^2}{2(n-1)} = -\infty$, so $\lim_{n\to\infty} (1-\tfrac{1}{n})^{n^2} = e^{-\infty} = 0.$

Hmm. nth term test for divergence doesn't work here.

① i) Second attempt. let's try RoT.

$$\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{1/n}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\frac{1}{e}<1$$ (see (1g) for the limit)

So by RoT the series converges.

j) $\lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}=\lim_{n\to\infty}\frac{2}{n+1}=0<1$, so the series converges by RaT.

k) $\lim_{n\to\infty}\frac{2^n}{n^2}\overset{LHR\times 2}{=}\frac{(\ln 2)^2 2^n}{2}=\infty>1$, so the series diverges by nth term test.

l) $\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(n+1)!}\cdot\frac{n!}{1\cdot 3\cdots(2n-1)}=\lim_{n\to\infty}\frac{2n+1}{n+1}=2>1$, series diverges by RaT.

m) $\lim_{n\to\infty}\left(\frac{(n!)^n}{(n^n)^2}\right)^{1/n}=\lim_{n\to\infty}\frac{n!}{n^2}=\lim_{n\to\infty}\frac{(n-1)!}{n}=\infty>1$ is pretty clear, so series diverges by RoT.

n) $\lim_{n\to\infty}\left(\frac{(n!)^n}{(n^{n^2})}\right)^{1/n}=\lim_{n\to\infty}\frac{n!}{n^n}=0<1$, so series converges by RoT.

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② a) $\lim_{n\to\infty}\frac{(n+1)^6}{6^{n+1}}\cdot\frac{6^n}{n^6}=\lim_{n\to\infty}\frac{1}{6}\frac{(n+1)^6}{n^6}=\cdots=\frac{1}{6}<1$, so converges by RaT.

b) don't split this to get $\infty-\infty$ - that's indeterminate. Instead, notice that
$\frac{1}{n}-\frac{1}{n^2}=\frac{1}{n}\left(1-\frac{1}{n}\right)$ is a lot like $\frac{1}{n}$ when n is large. so let's try LCT

$$\lim_{n\to\infty}\frac{\frac{1}{n}-\frac{1}{n^2}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{n}\left(1-\frac{1}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1.$$ since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (Harmonic),

$\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n^2}$ diverges by LCT (1 is positive and finite).

c) $\lim_{n\to\infty}\left(1-\frac{3}{n}\right)^n=\cdots=e^{-3}\neq 0$, so series diverges by nth term test.

d) $\lim_{n\to\infty}\frac{(n+2)(n+3)}{(n+1)!}\cdot\frac{n!}{(n+1)(n+2)}=\lim_{n\to\infty}\frac{n+3}{(n+1)(n+1)}=0$, so series converges by RaT.

1) e) $\sum_{n=1}^{\infty} \frac{n!\ln n}{n(n+2)!} = \sum_{n=1}^{\infty} \frac{\ln n}{n(n+2)(n+1)}$. Could try RaT, but LCT with $\frac{1}{n^2}$ might work.

$$\lim_{n\to\infty} \frac{\ln n}{n(n+2)(n+3)} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2 \ln n}{n(n+2)(n+3)} = \lim_{n\to\infty} \frac{n \ln n}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{n \ln n}{n^2+5n+6}$$

$$\overset{LHR}{=} \lim_{n\to\infty} \frac{n\cdot\frac{1}{n} + \ln n}{2n+5} = \lim_{n\to\infty} \frac{1+\ln n}{2n+5} \overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2} = 0,$$ so by LCT part 2,

$\sum_{n=1}^{\infty} \frac{n!\ln n}{n(n+2)!}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2>1$).

f) $\lim_{n\to\infty} \frac{n+5}{\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{(n+5)^2}{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{n^2+10n+5}{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{1+\frac{10}{n}+\frac{5}{n^2}}{1-1/n^2}} = \sqrt{1} = 1 \neq 0$

So series diverges by nth term test.

g) $\lim_{n\to\infty} \frac{1}{(2(n+1))!} \Big/ \frac{1}{(2n)!} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)(2n+1)2n!} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1.$

So series converges by RaT.

h) Use LCT with $\frac{1}{n^2}$ to show convergence (details omitted).

i) Use RaT to show convergence. (limit is $0<1$) (details omitted).

j) See worksheet 7 problem 4e) for almost identical problem. (diverges by IT)

K) $\lim_{n\to\infty} \left(\left(\frac{\ln n}{\ln n^2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{\ln n}{\ln n^2} = \lim_{n\to\infty} \frac{\ln n}{2\ln n} = \frac{1}{2} < 1$, so series converges by RT.

l) Use LCT with $\frac{1}{n}$ to show divergence (details omitted).

m) Use LCT with $1/n^{1/2}$ to show divergence (details omitted).

n) First, rewrite as $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^2 e^{-n}$. Then $\lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^2 e^{-(n+1)} \Big/ \left(\frac{n+1}{n}\right)^2 e^{-n} = \lim_{n\to\infty} \frac{(n+2)^2 n^2}{(n+1)^4 e}$

$=$ (choose your technique...) $= \frac{1}{e} < 1$, so series converges by RaT.

o) diverges by nth term test (limit should be obviously ∞.)

p) Use LCT with $\frac{1}{n^2}$ to show convergence (details omitted)

① q) Use nth term test to show divergence (details omitted)

r) use LCT with $\frac{1}{n^2}$ to show it converges

s) $\lim_{n\to\infty} \frac{6+7^n}{9+7^n} = \lim_{n\to\infty} \frac{\frac{6}{7^n}+1}{\frac{9}{7^n}+1} = 1$, so series diverges by nth term test.

t) use LCT with $1/n^{3/2}$ to show convergence.

u) use LCT with $1/n^2$ to show convergence.

v) use LCT with $1/n^5$ to show convergence.

w) use LCT with $1/n^{1/2}$ to show divergence

details omitted.

x) $S_1 = \sqrt{2}-\sqrt{1}$, $S_2 = \sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2} = \sqrt{3}-\sqrt{1}$, $S_3 = \sqrt{4}-\sqrt{3}+\sqrt{3}-\sqrt{1} = \sqrt{4}-\sqrt{1}$

$S_n = \sqrt{n+1}-1$. $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n+1}-1 = \infty$, so the series diverges.

y) This one is a bit tricky. it looks like it might converge, but it does not.
We use LCT with $1/n$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n\,n^{1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \lim_{n\to\infty} n^{-1/n}.$$

Now, $\lim_{n\to\infty} \ln(n^{-1/n}) = \lim_{n\to\infty} -\frac{1}{n}\ln n = \lim_{n\to\infty} -\frac{\ln n}{n} \overset{LHR}{=} \lim_{n\to\infty} -\frac{\frac{1}{n}}{1} = 0$

so $\lim_{n\to\infty} n^{-1/n} = e^0 = 1$. (so our original limit is 1). Thus, $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges by LCT since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

z) $S_1 = 1-\frac{1}{\sqrt{3}}$, $S_2 = 1-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{4}} = 1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}$, $S_3 = 1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{5}} = 1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{5}}$

$S_n = 1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}$. $\lim_{n\to\infty} S_n = 1+\frac{1}{\sqrt{2}}$, so the series converges.

(3) a) See worksheet 7, Problem 4f, it's the same.

b) Since $\ln n$ is not constant, we can't just compare to $1/n^3$, but we can use $1/n^2$

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{1/n^2} = \lim_{n\to\infty} \frac{n^2 \ln n}{n^3} = \lim_{n\to\infty} \frac{\ln n}{n} \overset{LHR}{=} \lim_{n\to\infty} \frac{1}{n} = 0,$$ so by $\underline{LCT}$ part 2,

$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $(p = 2 > 1)$.

c) This one is trickier. We can't compare to $1/n^2$. $1/n$ is also no good because this one probably converges. We need a power of n that grows faster than $\ln n$ to "subtract" from the bottom. Any positive power works. Let's try $1/n^{3/2}$.

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2} \ln n}{n^2} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} \overset{LHR}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n^{1/2}}} = \lim_{n\to\infty} \frac{2n^{1/2}}{n}$$

$= \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$. So by LCT case 2, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges since

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $(p = 3/2 > 1)$.

d) similar to c), but use $1/n^{5/4}$, for instance.