MATH 132 Worksheet 13 Taylor Series

1. Find Taylor/Maclaurin Series expansion for the following functions about the given point using the definition of Taylor series. (You may find your work on the previous worksheet helpful on some of these.)

   (a) $f(x) = e^x$ about $x = 1$

   (b) $f(x) = \sqrt{x+1}$ about $x = 0$

   (c) $f(x) = \frac{1}{1+x}$ about $x = 0$

   (d) $f(x) = \ln(x)$ about $x = 2$

   (e) $f(x) = \ln(x+1)$ about $x = 0$

   (f) $f(x) = \sin(x)$ about $x = \pi/2$

   (g) $f(x) = \cos(x)$ about $x = \pi/2$

2. Write down the Maclaurin series expansion for the following. (You should memorize these!) What is the interval of convergence for each of these?

   (a) $f(x) = \frac{1}{1-x}$ (b) $f(x) = \sin(x)$ (c) $f(x) = \cos(x)$ (d) $f(x) = e^x$

3. Using algebra/calculus and known Taylor series, find the Taylor Series expansion for the following functions about the given point.

   (a) $f(x) = \dfrac{x}{1-x}$ about $x = 0$

   (b) $f(x) = \dfrac{x^2}{1-x}$ about $x = 0$

   (c) $h(x) = \dfrac{1}{1-3x}$ centered at $x = 0$

   (d) $f(x) = e^{x^2}$ about $x = 0$

   (e) $f(x) = x\sin(x)$ about $x = 0$

   (f) $f(x) = e^{x-3}$ about $x = 0$

   (g) $f(x) = e^{5x^4}$ about $x = 0$

   (h) $f(x) = x^2\sin(3x)$ about $x = 0$

   (i) $f(x) = e^x$ about $x = 3$

   (j) $f(x) = \sin(x)$ about $x = \pi/2$

   (k) $f(x) = \cos(x)$ about $x = \pi/2$

① a) $f(x)=e^x$ about $x=1$. from WS 12 4a), $P_5(x)=e+e(x-1)+e/2\,(x-1)^2+e/3!\,(x-1)^3\ldots$

So it appears that $\boxed{e^x=\sum_{n=0}^{\infty} e\frac{(x-1)^n}{n!}=e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}}$

b) $f(x)=\sqrt{x+1}$ about $x=0$.

from WS12 4b), $P_5(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\frac{7}{256}x^5$.

Hmm. no obvious pattern here. $f^{(6)}(x)=\frac{945}{64}(x+1)^{-11/2}$, so $f^{(6)}(0)=\frac{945}{64}$, so the next term is

$\frac{945}{64\cdot 6!}x^6=\frac{21}{1024}x^6$. Still no obvious pattern. So let's just

say that $\boxed{\sqrt{x+1}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4-\frac{7}{256}x^5+\frac{21}{1024}x^6+\cdots}$

c) $f(x)=\frac{1}{1+x}$ about $x=0$. from WS12 (4d), $P_5(x)=1-x+x^2-x^3+x^4-x^5$.

so it's pretty clear $\boxed{\frac{1}{1+x}=\sum_{n=0}^{\infty}(-x)^n}$

d) $f(x)=\ln(x)$ about $x=2$. from WS 12 4e, we have

$P_5(x)=\ln 2+\frac{x-2}{2}-\frac{(x-2)^2}{2^2\cdot 2}+\frac{(x-2)^3}{2^3\cdot 3}-\frac{(x-2)^4}{2^4\cdot 4}+\frac{(x-2)^5}{2^5\cdot 5}$, so it's clear that

$$\boxed{\ln x=\ln 2+\sum_{n=1}^{\infty}\frac{(x-2)^n}{2^n\cdot n}(-1)^{n+1}}$$

e) $f(x)=\ln(x+1)$ about $x=0$. from WS 12 (4g), $P_5(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$

So clearly $\boxed{\ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}}$

① f) $f(x) = \sin x$ about $x = \pi/2$. From WS12 1b), $P_5(x) = 1 - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4$.

we can verify by computing 2 more derivatives that the coefficient of $(x-\pi/2)^5$ is 0 and of $(x-\pi/2)^6$ is $\frac{1}{6!}$. Now we see we only have even terms, so

we get $$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(x-\pi/2)^{2n}}{(2n)!}(-1)^n}$$

g) $f(x) = \cos x$ about $x = \pi/2$. Compute some derivatives, plug in $\pi/2$, and plug them into the formula to see that

$$\cos x = 0 - 1(x-\pi/2) + \frac{0}{2}(x-\tfrac{\pi}{2})^2 + \frac{(x-\pi/2)^3}{3!} + \frac{0}{4!}(x-\pi/2)^4 - \frac{(x-\pi/2)^5}{5!} + \dots$$

$$\boxed{\cos = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}}$$

② a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$

b) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x ($I = (-\infty, \infty)$)

c) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x

d) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

⇑
These 4 are in the textbook!
Just having you write them down helps you remember them.

3

a) $\frac{x}{1-x} = x \cdot \frac{1}{1-x} = x\sum_{n=0}^{\infty} x^n = \boxed{\sum_{n=0}^{\infty} x^{n+1}}$ or $\sum_{n=1}^{\infty} x^n$

b) $\frac{x^2}{1-x} = x^2 \frac{1}{1-x} = x^2 \sum_{n=0}^{\infty} x^n = \boxed{\sum_{n=0}^{\infty} x^{n+2}}$

c) $\boxed{\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n}$ or $\sum_{n=0}^{\infty} 3^n x^n$

d) $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$

e) $x\sin x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}}$

f) $e^{x-3} = \boxed{\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}}$ OR $e^{x-3} = e^x e^{-3} = \frac{1}{e^3} e^x = \frac{1}{e^3} \sum_{n=0}^{\infty} \frac{x^n}{n!}$

g) $e^{5x^4} = \sum_{n=0}^{\infty} \frac{(5x^4)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{5^n x^{4n}}{n!}}$

h) $x^2 \sin(3x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1} x^2}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+3}}{(2n+1)!}}$

i) using f), $e^x = e^3 e^{x-3} = e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{e^3 (x-3)^n}{n!}}$

j) $\sin x = \sin((x-\pi/2)+\pi/2) = \cos(x-\pi/2) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}}$

k) $\cos x = \cos((x-\pi/2)+\pi/2) = -\sin(x-\pi/2) = -\sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$