1. Find the area bounded by the given curves.

   (a) $y = x^2$ and $y = 2x - x^2$

   (b) $y = 2x$ and $y = x^2 - 4x$

   (c) $y = 3x^2, y = x^2 + 4$

   (d) $y = \sin(x), y = \cos(x), x = 0$, and $x = \pi/2$ (Be careful!)

   (e) $y = x - 1$ and $y^2 = 2x + 6$ (Read the problem carefully!)

   (f) $y = 1/x, y = 1/x^2, x = 2$.

   (g) $y = 5\cos x, \quad y = 5 - \frac{10x}{\pi}$

   (h) $y = 36 - x^2$ and $y = x^2 - 36$

2. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

   (a) $x + 2y = 2$, $x = 0$, $y = 0$; about the x-axis

   (b) $y = 3x^4$, $x = 1$, $x = -1$, $y = 0$; about the x-axis

   (c) $y = 3x^4$, $x = 1$, $y = 0$; about the y-axis

   (d) $y = 3x^4$, $x = 1$, $x = 0$, $y = 0$; about $x = 1$

   (e) $y = 3x^4$, $x = 1$, $x = 0$, $y = 0$; about $y = 3$

   (f) $y^2 = 4x$, $y = x$; about the x-axis

   (g) $y^2 = 4x$, $y = x$; about the y-axis

   (h) $y^2 = 4x$, $y = x$; about $x = 4$

   (i) $y^2 = 4x$, $y = x$; about $y = 4$

   (j) $x = e^{y^2}$, $y = 0$, $x = 0$, $y = 1$; about the x-axis

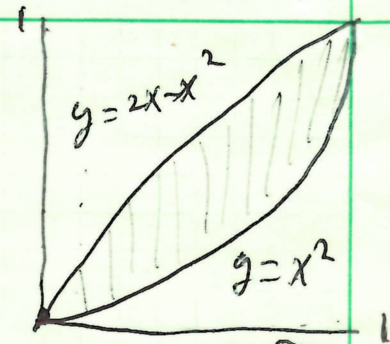
   (k) $y = \sin(x)$, $x = 0$, $x = \pi$, $y = 2$; about $y = 2$

① a) $y=x^2$, $y=2x-x^2$ Set equal: $x^2=2x-x^2$

$2x^2-2x=0$

$2x(x-1)=0$, $x=0,1$

$y=2x-x^2$ is above x^2 between 0 and 1, so



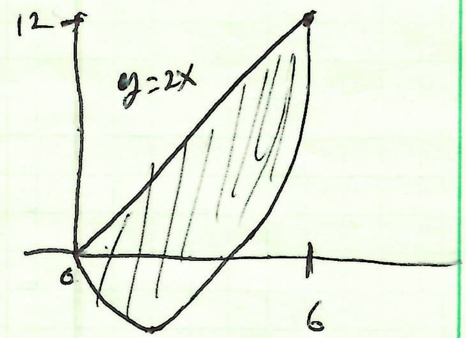
$$A=\int_0^1 (2x-x^2-x^2)\,dx=\int_0^1 (2x-2x^2)\,dx=x^2-\frac{2}{3}x^3\Big|_0^1=\left(1^2-\frac{2}{3}1^3\right)-(0-0)=1-\frac{2}{3}=\boxed{\frac{1}{3}}$$

b) $y=2x$, $y=x^2-4x$ Set equal: $2x=x^2-4x$

$x^2-6x=0$, $x(x-6)=0$

$x=0,6$

$y=2x$ is on top

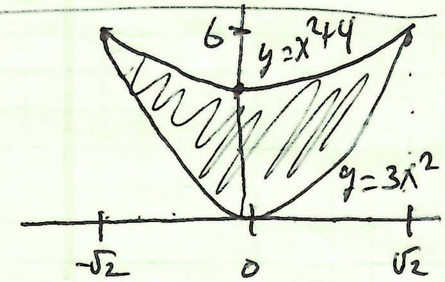


$$A=\int_0^6 2x-(x^2-4x)\,dx=\int_0^6(-x^2+6x)\,dx=-\frac{x^3}{3}+3x^2\Big|_0^6$$

$$=-\frac{6^3}{3}+3\cdot 6^2-(0-0)=-72+108=\boxed{36}$$

c) $y=3x^2$, $y=x^2+4$ Set equal: $3x^2=x^2+4$

$2x^2=4$, $x^2=2$, $x=\pm\sqrt{2}$

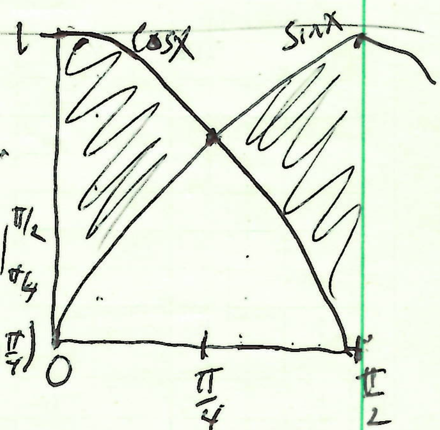


$$A=\int_{-\sqrt{2}}^{\sqrt{2}}(x^2+4-3x^2)\,dx=\int_{-\sqrt{2}}^{\sqrt{2}}-2x^2+4=-\frac{2}{3}x^3+4x\Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$=-\frac{2}{3}(\sqrt{2})^3+4\sqrt{2}-\left(-\frac{2}{3}(-\sqrt{2})^3+4(-\sqrt{2})\right)=-\frac{2}{3}\cdot 2\sqrt{2}+4\sqrt{2}-\frac{2}{3}\cdot 2\sqrt{2}+4\sqrt{2}$$

$$=-\frac{8}{3}\sqrt{2}+8\sqrt{2}=\boxed{\frac{16\sqrt{2}}{3}}$$

d) $y=\sin x$, $y=\cos x$, $x=0,\pi/2$. Note that $\sin x=\cos x$ when $x=\pi/4$ between 0 and $\pi/2$.



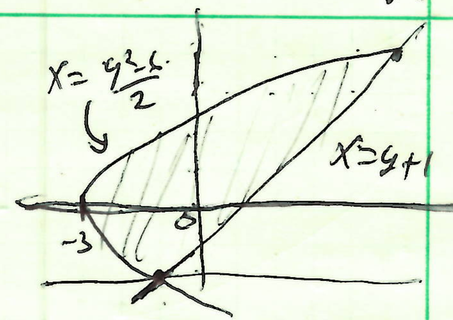
$$A=\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)\,dx=\sin x+\cos x\Big|_0^{\pi/4}+-\cos x-\sin x\Big|_{\pi/4}^{\pi/2}$$

$$=\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-(\sin 0+\cos 0)+\left(-\cos\frac{\pi}{2}-\sin\frac{\pi}{2}\right)-\left(-\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-(0+1)+(-1+0)-\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)=\boxed{2\sqrt{2}-2}$$

① e) $y = x-1$ and $y^2 = 2x+6$,

or $x = y+1$ and $x = \frac{y^2-6}{2}$



we need to do this with respect to y.

$y+1 = \frac{y^2-6}{2}$, $2y+2 = y^2-6$, $y^2-2y-8=0$, $(y-4)(y+2)=0$, $y=4,-2$

graph is sideways – the curve on the right is on top. so

$$A = \int_{-2}^{4} y+1-\left(\frac{y^2-6}{2}\right)dy = \int_{-2}^{4} -\frac{y^2}{2}+y+4\,dy = -\frac{y^3}{6}+\frac{y^2}{2}+4y\Big|_{-2}^{4}$$

$$= \frac{-4^3}{6}+\frac{4^2}{2}+4\cdot 4-\left(\frac{-(-2)^3}{6}+\frac{(-2)^2}{2}+4(-2)\right) = -\frac{32}{3}+8+16-\frac{4}{3}-2+8 = \boxed{18}$$

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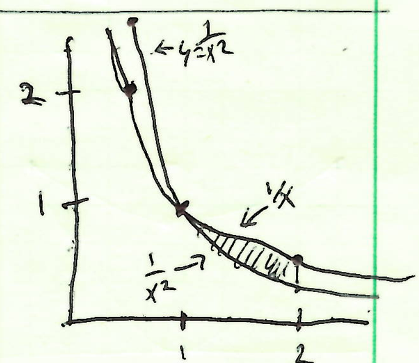
f) $y = 1/x$, $y = 1/x^2$, $x=2$ clearly they intersect $x=1$



$$A = \int_1^2 \frac{1}{x}-\frac{1}{x^2}dx = \ln x + \frac{1}{x}\Big|_1^2 = \ln 2 + \frac{1}{2} - (\ln 1 + 1)$$

$$= \ln 2 + \tfrac{1}{2} - 1 = \boxed{\ln 2 - \tfrac{1}{2}}$$

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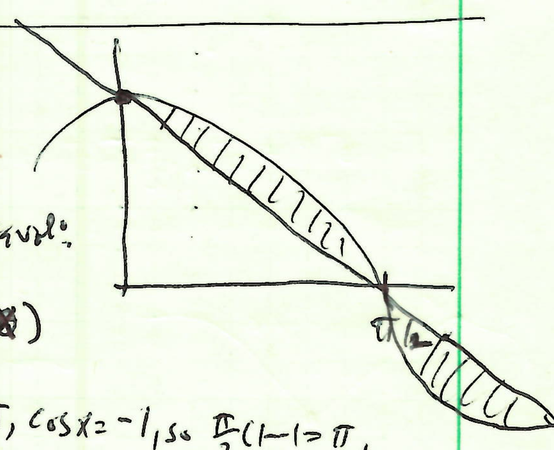
g) $y = 5\cos x$, $y = 5 - \frac{10x}{\pi}$

Need to graph these to determine intersection.



3rd point unclear from graph. let's try setting them equal:

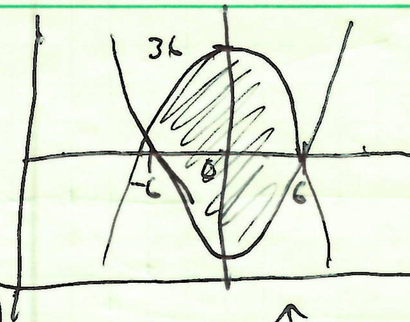
$5\cos x = 5 - \frac{10x}{\pi}$, $\pi\cos x = \pi - 2x$ $\quad x = \frac{\pi}{2}(1-\cos x)$

we can see if $x=0, \pi/2$ we have solutions. when $x=\pi$, $\cos x = -1$, so $\frac{\pi}{2}(1-(-1)) = \pi$,

so that is intersection as well. finally:

$$A = \int_0^{\pi/2} 5\cos x - \left(5-\frac{10x}{\pi}\right)dx + \int_{\pi/2}^{\pi}\left(5-\frac{10x}{\pi}\right) - 5\cos x\,dx = \text{lots of skipped steps} = \boxed{10-\frac{5\pi}{2}}$$

① h) $y=36-x^2$, $y=x^2-36$, clearly intersect when $x=6,-6$



$$A=\int_{-6}^{6}\left[36-x^2-(x^2-36)\right]dx=\int_{-6}^{6}(72-2x^2)\,dx$$

$$=72x-\tfrac{2}{3}x^3\Big|_{-6}^{6}=72\cdot6-\tfrac{2}{3}6^3-\left(72(-6)-\tfrac{2}{3}(-6)^3\right)$$

$$=432-144+432-144=\boxed{576}$$

not to scale.

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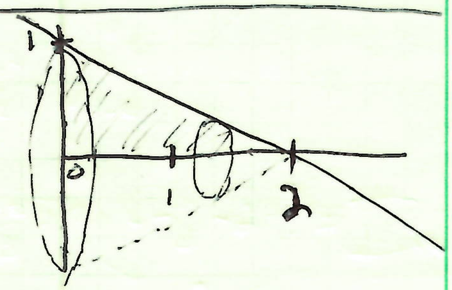
② a) $x+2y=2$, $x=0$, $y=0$; about x-axis



$y=-\frac{x}{2}+1$ $R=-\frac{x}{2}+1$ →

$$V=\int_0^2\pi\left(-\tfrac{x}{2}+1\right)^2dx=\int_0^2\pi\left(\tfrac{x^2}{4}-x+1\right)dx$$

$$=\pi\left[\tfrac{x^3}{12}-\tfrac{x^2}{2}+x\right|_0^2=\pi\left(\tfrac{8}{12}-\tfrac{4}{2}+2\right)=\boxed{\tfrac{2}{3}\pi}$$

note: matches formula for volume of a cone.

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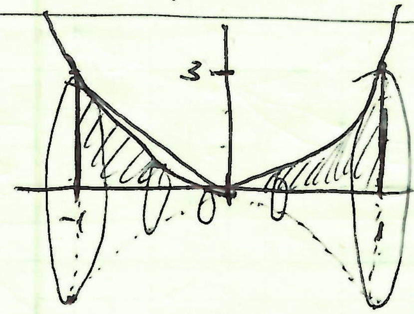
b) $y=3x^4$, $x=1$, $x=-1$, $y=0$; about x-axis



$$V=\int_{-1}^{1}\pi(3x^4)^2dx=\pi\int_{-1}^{1}9x^8dx=\pi x^9\Big|_{-1}^{1}=\pi(1--1)$$

$$=\boxed{2\pi}$$

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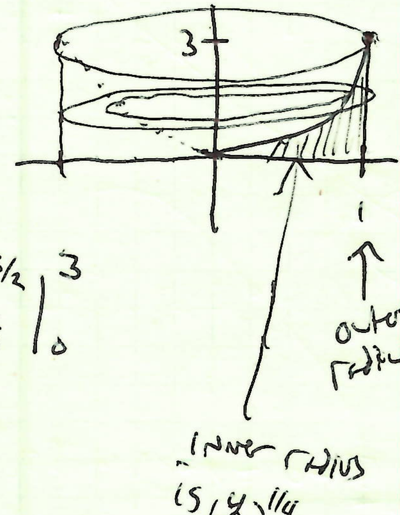
c) $y=3x^4$, $x=1$, $y=0$, about $\boxed{y\text{-axis}}$ ← new!

Washers!

Washers are along y so integrate along y, so resolve:

$x^4=\frac{y}{3}$, $x=\left(\frac{y}{3}\right)^{1/4}$ ← inner. outer is $x=1$



$$V=\int_0^3\pi\left(1^2-\left(\left(\tfrac{y}{3}\right)^{1/4}\right)^2\right)dy=\int_0^3\pi\left(1-\tfrac{\sqrt{y}}{\sqrt{3}}\right)dy=\pi\left(y-\tfrac{2y^{3/2}}{3\sqrt{3}}\right)\Big|_0^3$$

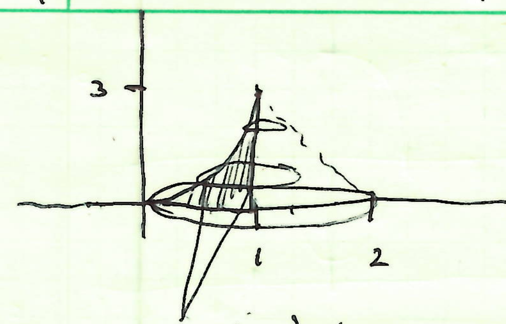
$$=\pi\left[\left(3-\tfrac{2\,3^{3/2}}{3\sqrt{3}}\right)-0\right]=\pi(3-2)=\boxed{\pi}$$

② d) $y=3x^4$, $x=1, x=0, y=0$, about $x=1$ ← new!

disc are along y-axis, so we integrate w.r.t. y.

$x=\left(\frac{y}{3}\right)^{1/4}$ (see 2c)

be careful here $R=1-\left(\frac{y}{3}\right)^{1/4}$.



R is this distance, so $1-\left(\frac{y}{3}\right)^{1/4}$

so $V=\int_0^3 \pi\left(1-\left(\frac{y}{3}\right)^{1/4}\right)^2 dy = \pi\int_0^3\left(1-2\left(\frac{y}{3}\right)^{1/4}+\frac{\sqrt{y}}{\sqrt{3}}\right)dy$

$$=\pi\left[y-\frac{8}{5}\frac{y^{5/4}}{3^{1/4}}+\frac{2y^{3/2}}{3\sqrt{3}}\right]_0^3=\pi\left[3-\frac{8}{5}\frac{3^{5/4}}{3^{1/4}}+2\frac{3^{3/2}}{3\sqrt{3}}-(0)\right]$$

$$=\pi\left[3-\frac{8\cdot 3}{5}+2\right]=\frac{15-24+10}{5}\pi=\boxed{\frac{\pi}{5}}$$

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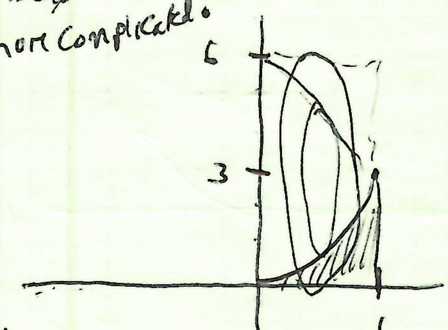
e) $y=3x^4$, $x=1$, $x=0$, $y=0$, about $y=3$

↑ makes radius more complicated!

outer radius is 3.

inner radius is $3-3x^4$ (do you see why?)



so

$$V=\int_0^1\pi\left(3^2-(3-3x^4)^2\right)dx=\pi\int_0^1 9-9+18x^4-9x^8\,dx$$

$$=\int_0^1\pi(18x^4-9x^8)dx=\pi\left[\frac{18}{5}x^5-x^9\right]_0^1=\pi\left[\left(\frac{18}{5}-1\right)-(0)\right]=\boxed{\frac{13}{5}\pi}$$

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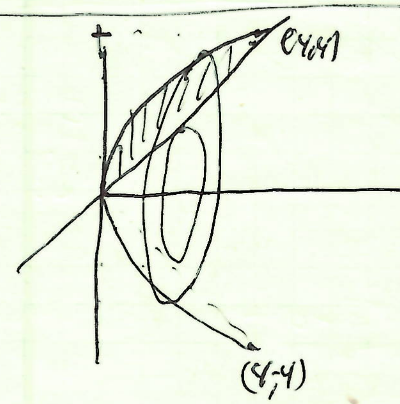
f) $y^2=4x$, $y=x$, about x-axis, so $y=\pm 2\sqrt{x}$



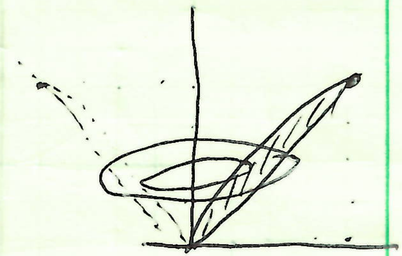
$$V=\int_0^4\pi\left[(2\sqrt{x})^2-x^2\right]dx=\pi\int_0^4(4x-x^2)dx$$

$$=\pi\left[2x^2-\frac{x^3}{3}\Big|_0^4\right]=\pi\left[2\cdot 4^2-\frac{4^3}{3}-0\right]$$

$$=\boxed{\frac{32}{3}\pi}$$

(2) g) $y^2 = 4x$, $y = x$, about y-axis

integrate with respect to y! $x = \frac{y^2}{4}$, $x = y$



$$V = \pi \int_0^4 \left[y^2 - \left(\frac{y^2}{4}\right)^2 \right] dy = \pi \int_0^4 \left(y^2 - \frac{y^4}{16}\right) dy$$

$$= \pi \left[\frac{y^3}{3} - \frac{y^5}{5 \cdot 16} \right|_0^4 = \pi \left(\left(\frac{4^3}{3} - \frac{4^5}{5 \cdot 16} \right) - 0 \right) = \pi \left[\frac{64}{3} - \frac{64}{5} \right] = \boxed{\frac{128}{15}\pi}$$

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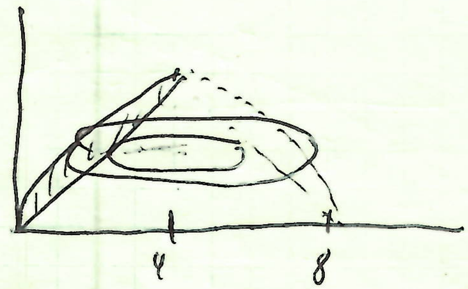
h) $y^2 = 4x$, $y = x$, about $x = 4$

integrate with respect to y. $x = y$, $x = \frac{y^2}{4}$



outer radius: $4 - \frac{y^2}{4}$

inner radius: $4 - y$

So

$$V = \int_0^4 \pi \left[\left(4 - \frac{y^2}{4}\right)^2 - (4 - y)^2 \right] dy =$$

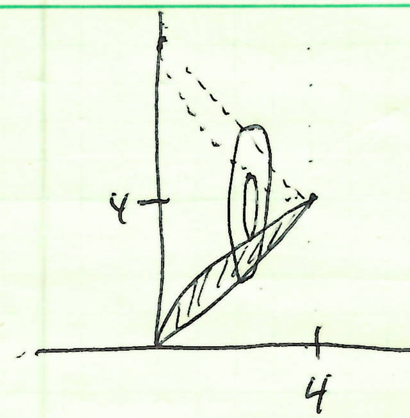
$$= \pi \int_0^\pi \left(\cancel{16} - 2y^2 + \frac{y^4}{16} - \cancel{16} + 8y - y^2\right) dy = \pi \int_0^\pi \frac{y^4}{16} + 8y - 3y^2 \, dy$$

$$= \pi \left[\frac{y^5}{16 \cdot 5} + 4y^2 - y^3 \right|_0^4 = \pi \left[\left[\frac{4^5}{16 \cdot 5} - 4 \cdot 4^2 - 4^3 \right] - 0 \right] = \boxed{\frac{64\pi}{5}}$$

(2) c) $y^2 = 4x$, $y = x$, about $y = 4$

$y = \pm\sqrt{2}x$, we want $y = \sqrt{2}x$

outer radius is $4 - x$, inner radius is $4 - 2\sqrt{x}$



$$V = \pi\int_0^4 (4-x)^2 - (4 - 2\sqrt{x})^2 \, dx$$

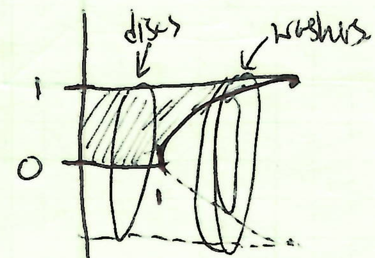
$$= \pi\int_0^4 (16 - 8x + x^2 - 16 + 16\sqrt{x} - 4x)\,dx = \pi\int_0^4 (x^2 - 12x + 16\sqrt{x})\,dx$$

$$= \pi\left[\frac{x^3}{3} - 6x^2 + \frac{32}{3}x^{3/2}\right]_0^4 = \pi\left[\frac{4^3}{3} - 6\cdot 4^2 + \frac{32\cdot 4^{3/2}}{3} - [0]\right] = \cdots = \boxed{\frac{32}{3}\pi}$$

j) $x = e^{y^2}$, $y = 0$, $x = 0$, $y = 1$, about x-axis.

$\ln x = \ln e^{y^2} = y^2 \ln e = y^2$, so $\ln x = y^2$

so $y = \sqrt{\ln x}$.



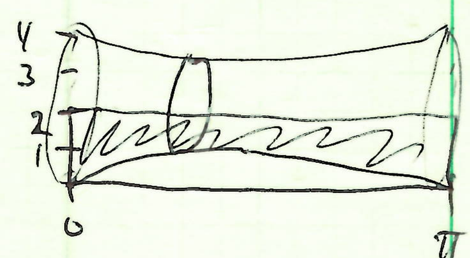
$$\text{so } V = \int_0^1 \pi\cdot 1^2 + \int_1^e \pi\left[1^2 - (\sqrt{\ln x})^2\right]dx = \pi x\Big|_0^1 + \pi\int_1^e (1 - \ln x)\,dx$$

$$= \pi + \pi\,(x - (x\ln x - x))\Big|_1^e = \pi + \pi\left[2e - e\ln e - (2\cdot 1 - 1\ln 1)\right] = \cdots = (e-1)\pi$$

* This is much easier with shells!

k) $y = \sin x$, $x = 0$, $x = \pi$, $y = 2$, about $y = 2$

Radius is $2 - \sin x$.



$$V = \int_0^\pi \pi(2 - \sin x)^2\,dx = \pi\int_0^\pi 4 - 4\sin x + \sin^2 x\,dx$$

$$= \pi\left(4x + 4\cos x + \tfrac{1}{2} - \tfrac{1}{4}\sin 2x\right)\Big|_0^\pi = \cdots = \boxed{\pi\left[\frac{9}{2}\pi - 8\right]}$$