1. Set up integrals (both washers and shells, if possible) that can be used to find the volume of the solid obtained by rotating the region bounded by the curves about the given line:

   (a) $y = 7x^2$, $x = 1$, and $y = 0$, about the x-axis

   (b) $f(x) = e^x$ and $y = 0$, about the x-axis over the interval $[0, 2]$

   (c) $y = 18 - x$, $y = 3x - 6$ and $x = 0$, about the y-axis

   (d) $y = x^2$ and $y = 4x$, about the line $x = 4$

   (e) $y = x^2$ and $y = 5x$, about the line $y = 0$

   (f) $x = 0$, $x = 1$, $y = 0$, and $y = 3 + x^7$, about the x-axis

   (g) $y = x^2$, $x = 3$, and $y = 0$, about the x-axis

   (h) $x = 5y$ and $y^3 = x$ (with $y \geq 0$), about the y-axis

   (i) $y = x^2$ and $y = 1$, about the line $y = 2$

   (j) $x = y^2$ and $x = 2y$, about the line $y = 2$

   (k) $x = y^2$ and $x = 1$, about the line $x = 2$

   (l) $y = x$ and $y = \sqrt{x}$, about the line $y = 1$

2. Find the length of the curves

   (a) $y = x^{1/2} - \dfrac{1}{3}x^{3/2}$ for $1 \leq x \leq 4$

   (b) $y = x^2 - (\ln(x))/8$ for $1 \leq x \leq 2$

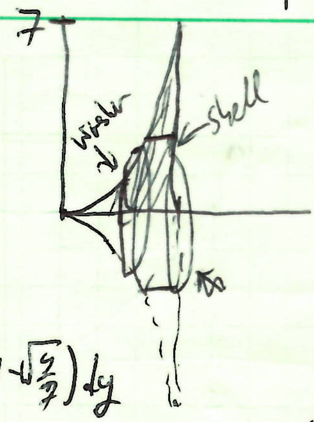
3. Find the areas of the surfaces generated by revolving the curves about the indicated axes.

   (a) $y = x^3/9$, $0 \leq x \leq 2$, about the x-axis

   (b) $y = \sqrt{x}$, $3/4 \leq x \leq 15/4$, about the x-axis

   (c) $y = x^{1/2} - \frac{1}{3}x^{3/2}$, $1 \leq x \leq 4$, about the y-axis

① a) $y=7x^2$, $x=1$, $y=0$, about x-axis

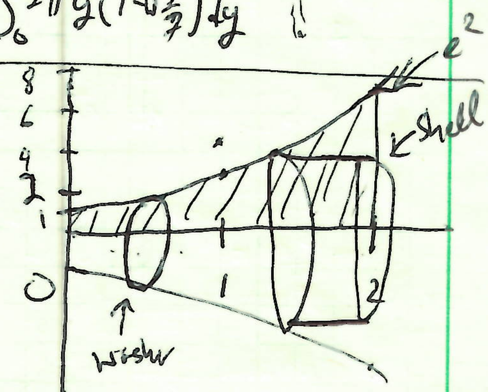


$R=y=\cancel{7}x^2$. Washer: $\int_0^1 \pi(7x^2)^2\,dx = \pi\int_0^1 49x^4\,dx$

Shells: $h=1-x$, $y=7x^2 \Rightarrow x^2=y/7$, $x=\sqrt{y/7}$, ~~...~~

so $h=1-\sqrt{y/7}$. $R=y$ so $V=\int_0^7 2\pi rh\,dy = \int_0^7 2\pi y\left(1-\sqrt{\frac{y}{7}}\right)dy$

b) $f(x)=e^x$, $y=0$, over $[0,2]$, about x-axis



Washers: $R=e^x$, so $V=\int_0^2 \pi(e^x)^2\,dx = \int_0^2 \pi e^{2x}\,dx$

shell: $R=e^x=y$, $h=2-x$, where $y=e^x$, so $x=\ln y$

so $h=2-\ln y$. $V=\int_1^{e^2} 2\pi y(2-\ln y)\,dy$. But that misses the cylinder of radius 1 about the x-axis. it has volume 2π ($h=2$, $R=1$, $V=\pi R^2 h$)

So $V=\int_1^{e^2} 2\pi y(2-\ln y)\,dy + 2\pi$

Important notes

① Shells and washers are always with respect to opposite axes.

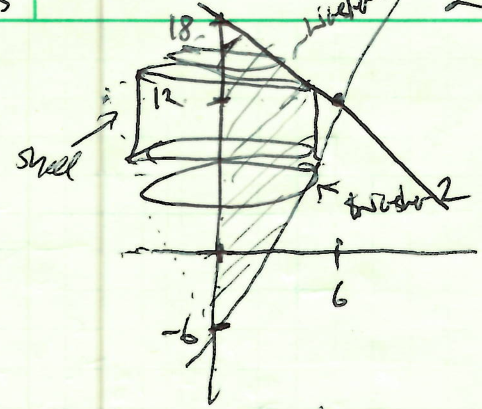
② shells and washers usually have different limits of integration since they are along different axes

③ Shells has limits that start at the axis of rotation and go out ONE direction in other words, only think about one side of the drawing when thinking about the limits of integration. for instance, if 1(b) the limits were from $-e^2$ to e^2, we would be double counting shells and/or trying to think of shells that don't exist [e.g. the curve doesn't exist for negative y values, and between 0 and 1 it isn't what we want.]

④ The axis of rotation determines whether you integrate with respect to x or y for each method.

① c) $y=18-x$, $y=3x-6$, $x=0$ about y-axis

$18-x=3x-6 \iff 4x=24 \iff x=6$, so they intersect @ $x=6$, where $y=12$. @ $x=0$, one has value 18, the other -6. so we graph: →
[graph, as most are, is not to scale



With washers, there are 2 regions. We integrate with respect to y, so need to solve equations for y: $y=18-x \Rightarrow x=18-y$ and $y=3x-6 \Rightarrow x=\frac{y+6}{3}$.

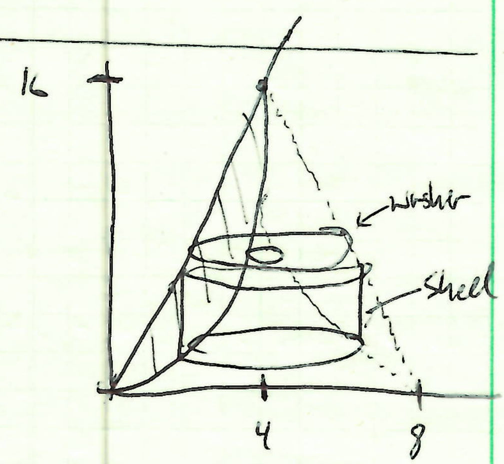
In both cases the radius is the value of the function. Thus:

Washers $$V=\int_{12}^{18}\pi(18-y)^2\,dy+\int_{-6}^{12}\pi\left(\frac{y+6}{3}\right)^2dy$$

Shells: $h=18-x-(3x-6)=24-4x$. $r=x$, so

$$V=\int_0^6 2\pi x(24-4x)\,dx=\int_0^6 2\pi(24x-4x^2)\,dx.$$

d) $y=x^2$, $y=4x$, about $x=4$



Washers: with respect to y. $y=x^2 \Rightarrow x=\sqrt{y}$. $y=4x \Rightarrow x=\frac{y}{4}$

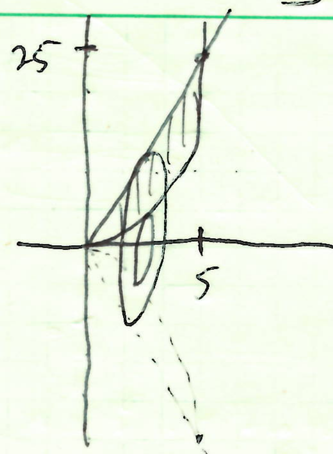
outer radius is $4-\frac{y}{4}$. inner radius is $4-\sqrt{y}$.

$$V=\int_0^{16}\pi\left[\left(4-\frac{y}{4}\right)^2-\left(4-\sqrt{y}\right)^2\right]dy$$

Shells: with respect to x. height is $4x-x^2$. radius is $4-x$.

$$V=\int_0^4 2\pi rh\,dx=\int_0^4 2\pi(4-x)(4x-x^2)\,dx$$

① e) $y=x^2$, $y=5x$ about $y=0$



Washers: $\displaystyle\int_0^5 \pi\left((5x)^2-(x^2)^2\right)dx = \pi\int_0^5 (25x^2-x^4)\,dx$

Shells: $x=\sqrt{y}$, $x=y/5$. $h=\sqrt{y}-y/5$, $R=y$

$$V=\int_0^{25} 2\pi Rh\,dy=\int_0^{25} 2\pi y\left(\sqrt{y}-y/5\right)dy$$

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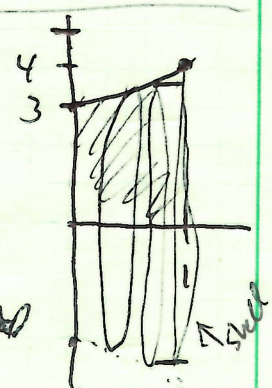
f) $x=0$, $x=1$, $y=0$, $y=3+x^7$ about x-axis



Washers: Just discs, so $V=\int_0^1 \pi(3+x^7)^2\,dx$.

Shells: More complicated, a cylinder of radius 3 and height 1 surrounded by shells from 3 to 4. height is $1-x$, but we need to do it with respect to y. $y=3+x^7 \Rightarrow x^7=y-3 \Rightarrow x=(y-3)^{1/7}$

so $h=1-(y-3)^{1/7}$, $r=y$. so

cylinder

$$V=\int_3^4 2\pi y\left(1-(y-3)^{1/7}\right)dy+3^2\cdot 1\cdot\pi$$

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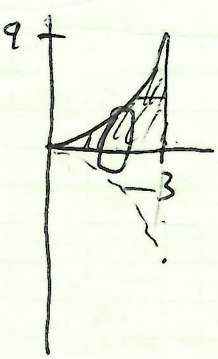
g) $y=x^2$, $x=3$, $y=0$ about x-axis $\qquad x=\sqrt{y}$



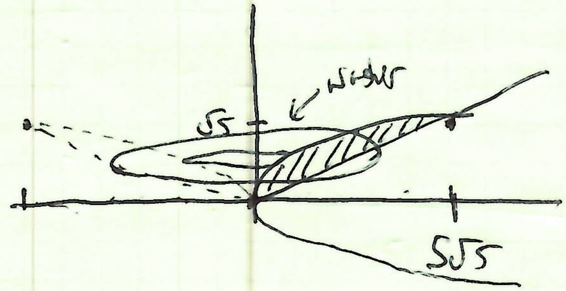
Washers: $V=\int_0^3 \pi(x^2)^2\,dx$

Shells: $V=\int_0^9 2\pi y(3-\sqrt{y})\,dy$

① h) $x = 5y$ and $y^3 = x$ (with $y \geq 0$), about y-axis

where do they intersect? $y^3 = 5y$, $y^3 - 5y = 0$, $y(y-\sqrt{5})(y+\sqrt{5}) = 0$, so $y = 0, \sqrt{5}, -\sqrt{5}$.

we only care about $y = 0, \sqrt{5}$.

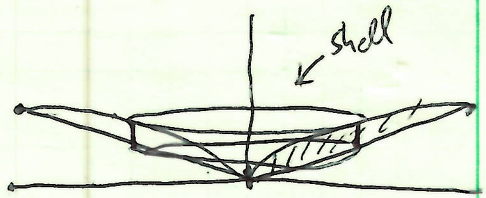


Washers:

$$V = \int_0^{\sqrt{5}} \pi\left((5y)^2 - (y^3)^2\right) dy$$

Shells: with respect to x, so $x = 5y \Rightarrow y = \frac{x}{5}$, $y^3 = x \Rightarrow x = \sqrt[3]{x}$.

radius is just x, height is $\sqrt[3]{x} - \frac{x}{5}$, so



$$V = \int_0^{5\sqrt{5}} 2\pi x\left(\sqrt[3]{x} - \frac{x}{5}\right) dx$$

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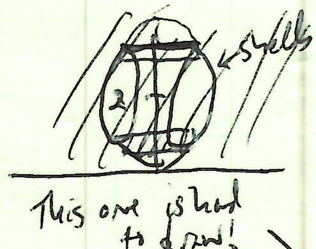
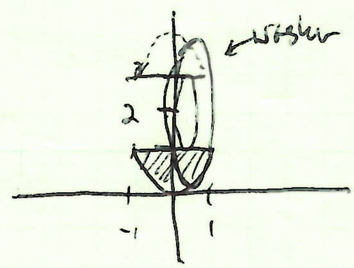
i) $y = x^2$, $y = 1$, about $y = 2$



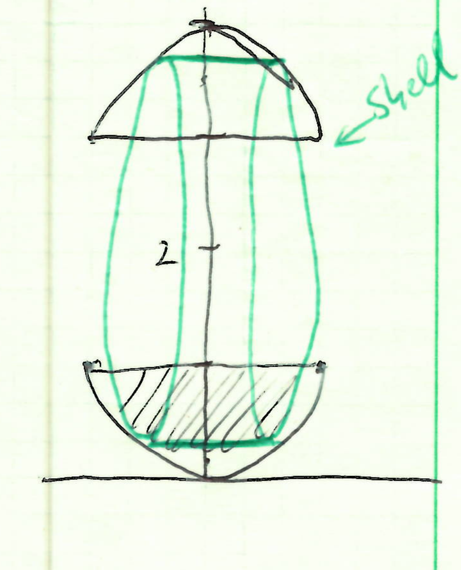


Washers:

$$V = \int_{-1}^{1} \pi\left[(2-x^2)^2 - 1^2\right] dx$$

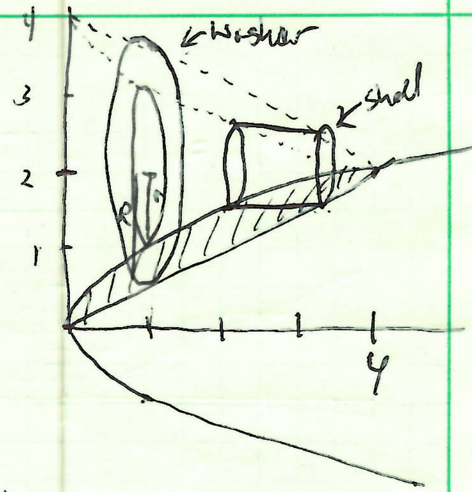
Shells: $x = \pm\sqrt{y}$, $h = \sqrt{y} - -\sqrt{y} = 2\sqrt{y}$, $r = 2 - y$ ← tricky!

$$V = \int_0^1 2\pi\, 2\sqrt{y}\,(2-y)\, dy$$



① j) $x=y^2$, $x=2y$, about $y=2$; $(0,0)$, $(4,2)$

$y^2=2y$, $y^2-2y=0$, $y(y-2)=0$, $y=0,2$

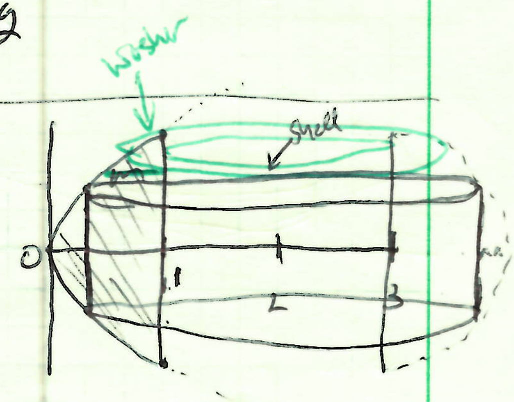


~~Washers~~: $y=\sqrt{x}$, $y=\frac{x}{2}$, outer $R=2-\frac{x}{2}$, inner $R=2-\sqrt{x}$

$$V=\int_0^4 \pi\left[\left(2-\frac{x}{2}\right)^2-\left(2-\sqrt{x}\right)^2\right]dx$$

Shells: $h=2y-y^2$, $r=2-y$, $V=\int_0^2 2\pi(2y-y^2)(2-y)\,dy$

k) $x=y^2$, $x=1$, about $x=2$

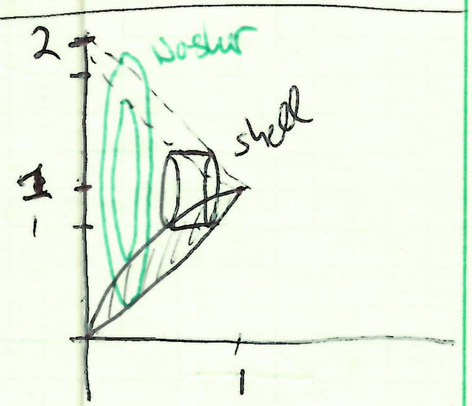


Washers: $\int_{-1}^{1}\pi\left[(2-y^2)^2-1^2\right]dy$

Shells: $h=2\sqrt{x}$, $r=2-x$, $V=\int_0^1 2\pi\, 2\sqrt{x}\,(2-x)\,dx$

[note: This is identical to i) except x and y are swapped.]

l) $y=x$, $y=\sqrt{x}$, about $y=1$



~~Washers:~~ $\cancel{\int_0^1 \pi\left[(1-\sqrt{x})^2-(1-x)^2\right]dx}$ oops — backwards!

Shells: $x=y$, $x=y^2$, $h=y-y^2$, $r=1-y$

$$\int_0^1 2\pi(y-y^2)(1-y)\,dy$$

Washers: $\int_0^1 \pi\left[(1-x)^2-(1-\sqrt{x})^2\right]dx$

(2) a) length of curve $y = x^{1/2} - \frac{1}{3}x^{3/2}$ for $1 \le x \le 4$

we first need:

$$y' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) = \frac{1}{2}\left(\frac{1-x}{\sqrt{x}}\right)$$ ← we'll find this form helpful.

Then

$$A.L. = \int_1^4 \sqrt{1 + \left[\frac{1}{2}\left(\frac{1-x}{\sqrt{x}}\right)\right]^2}\, dx = \int_1^4 \sqrt{1 + \frac{(1-x)^2}{4x}}\, dx$$

$$= \int_1^4 \sqrt{\frac{4x + 1 - 2x + x^2}{4x}}\, dx = \int_1^4 \sqrt{\frac{x^2 + 2x + 1}{4x}}\, dx = \int_1^4 \sqrt{\frac{(x+1)^2}{4x}}\, dx$$

$$= \int_1^4 \frac{x+1}{2\sqrt{x}}\, dx = \int_1^4 \frac{\sqrt{x}}{2} + \frac{1}{2\sqrt{x}}\, dx = \frac{1}{3}x^{3/2} + x^{1/2}\Big|_1^4$$

$$= \frac{1}{3}4^{3/2} + 4^{1/2} - \left(\frac{1}{3}\cdot 1 + 1\right) = \frac{8}{3} + 2 - \frac{1}{3} - 1 = \boxed{\frac{10}{3}}$$

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b) length of $y = x^2 - (\ln x/8)$ for $1 \le x \le 2$. $y' = 2x - \frac{1}{8x} = \frac{16x^2 - 1}{8x}$

$$1 + (y')^2 = 1 + \left(\frac{16x^2 - 1}{8x}\right)^2 = 1 + \frac{16^2x^4 - 32x^2 + 1}{64x^2} = \frac{64x^2 + 16^2x^4 - 32x^2 + 1}{64x^2}$$

$$= \frac{16^2x^4 + 32x^2 + 1}{64x^2} = \frac{(16x^2+1)^2}{(8x)^2}, \text{ so}$$

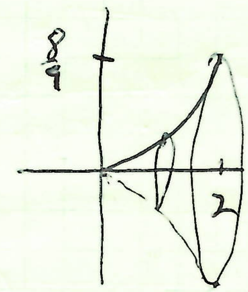
$$AL = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \sqrt{\frac{(16x^2+1)^2}{(8x)^2}}\, dx = \int_1^2 \frac{(16x^2+1)}{8x}\, dx$$

$$= \int_1^2 2x + \frac{1}{8x}\, dx = x^2 + \frac{\ln x}{8}\Big|_1^2 = \left(2^2 + \frac{\ln 2}{8}\right) - \left(1^2 + \frac{\ln 1}{8}\right)$$

$$= 4 + \frac{\ln 2}{8} - (1 + 0) = \boxed{3 + \frac{\ln 2}{8}}$$

③ a) Surface area of $y = x^3/9$, $0 \le x \le 2$, revolved around x-axis. $y' = x^2/3$

about x-axis, so $SA = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$



$$= \int_0^2 2\pi \frac{x^3}{9}\sqrt{1+\left(\frac{x^2}{3}\right)^2}\,dx = \int_0^2 \frac{2\pi}{9}x^3\sqrt{1+\frac{x^4}{9}}\,dx$$

$u = 1 + \frac{x^4}{9}$ $\to x=0\ u=1$, $x=2,\ u = 1+\frac{16}{9} = 25/9$

$du = \frac{4}{9}x^3\,dx$ $\quad \frac{du}{2} = \frac{2}{9}x^3\,dx$

$$= \int_1^{25/9} \frac{\pi}{2}u^{1/2}\,du = \frac{\pi}{2}\frac{2}{3}u^{3/2}\Big|_1^{25/9} = \frac{\pi}{3}\left[\left(\frac{25}{9}\right)^{3/2} - 1^{3/2}\right] = \frac{\pi}{3}\left[\left(\frac{5}{3}\right)^3 - 1\right]$$

$$= \frac{\pi}{3}\left[\frac{125}{27} - \frac{27}{27}\right] = \boxed{\frac{98}{81}\pi}$$

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b) Surface area of $y = \sqrt{x}$, $3/4 \le x \le 15/4$, about x-axis

$SA = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$ since rotation about x-axis. $y' = \frac{1}{2\sqrt{x}}$ $(y')^2 = \frac{1}{4x}$

$$= \int_{3/4}^{15/4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = \int_{3/4}^{15/4} 2\pi\sqrt{\frac{4x^2+x}{4x}}\,dx$$

$$= \int_{3/4}^{15/4} 2\pi\sqrt{\frac{4x+1}{4}}\,dx = \int_{3/4}^{15/4} \pi\sqrt{4x+1}\,dx = \frac{2\pi}{3\cdot 4}(4x+1)^{3/2}\Big|_{3/4}^{15/4}$$

$$= \frac{\pi}{6}\left[16^{3/2} - 4^{3/2}\right] = \frac{\pi}{6}\left[4^3 - 2^3\right] = \frac{\pi}{6}\left[64 - 8\right] = \frac{56}{6}\pi = \boxed{\frac{28}{3}\pi}$$

③ c) $y = x^{1/2} - \frac{1}{3}x^{3/2}$, $1 \le x \le 4$, rotate around y-axis.

Other formula: $SA = \int_a^b 2\pi x \sqrt{1 + f'(x)^2}\, dx$

$f'(x) = y' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)$

$f'(x)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)$

$f'(x)^2 + 1 = \frac{1}{4x} - \frac{1}{2} + \frac{x}{4} + 1 = \frac{1}{4x} + \frac{1}{2} + \frac{x}{4} = \frac{1 + 2x + x^2}{4x} = \frac{(1+x)^2}{4x}$

so

$$SA = \int_1^4 2\pi x \sqrt{\frac{(1+x)^2}{4x}}\, dx = \int_1^4 2\pi x \frac{(1+x)}{2\sqrt{x}}\, dx$$

$$= \int_1^4 \frac{2\pi}{2} \sqrt{x}(1+x)\, dx = \int_1^4 \pi\left(x^{1/2} + x^{3/2}\right) dx = \pi\left(\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2}\right)\Big|_1^4$$

$$= \pi\left[\left(\frac{2}{3}4^{3/2} + \frac{2}{5}4^{5/2}\right) - \left(\frac{2}{3}1^{3/2} + \frac{2}{5}1^{5/2}\right)\right] = \pi\left[\frac{2}{3}\cdot 2^3 + \frac{2}{5}2^5 - \frac{2}{3} - \frac{2}{5}\right]$$

$$= \pi\left[\frac{16}{3} + \frac{64}{5} - \frac{2}{3} - \frac{2}{5}\right] = \pi\left[\frac{16\cdot 5 + 64\cdot 3 - 2\cdot 5 - 2\cdot 3}{3\cdot 5}\right] = \boxed{\frac{256}{15}\pi}$$