1. A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed. The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and the rope?

2. How much work is performed pulling a 60 feet rope weighing 66 lbs/ft up a cliff face? At what point has half of the work been completed?

3. A cable weighing 6 lb/ft is connected to a construction elevator weighing 1500 lbs. Find the work done in lifting the elevator to a height of 500 ft.

4. A conical tank of height 20 feet and radius 8 feet, with the point at the top, is filled up to 17 feet with Dr. Pepper (63 lb/ft^3). Compute the amount of work required to pump all of the Dr. Pepper to a height of 3 feet above the top of the tank.

5. Redo the previous problem but with the tank flipped so it has the point at the bottom. Before doing it, think about which one should take more work. Do your answers support your guess?

6. A cylindrical tank of diameter 7 feet and height 13 feet is filled up to 11 feet with lead (58.9 lb/ft^3). Compute the amount of work required to pump all of the lead to a height of 5 feet above the top of the tank.

7. A hemispherical tank (with flat side up) of radius 20 ft is filled with water (weighing about 62.4 lb/ft^3) to a 15 ft depth. Find the work done in pumping all the water to the top of the tank.

8. Refer to previous problem. Find the work done in pumping all the water to a point 10 ft above the hemispherical tank.

① Let x be the amount of rope pulled in. Then the force on the rope is $(20-x)*0.08$ lb. The force on the bucket is always 5 lb.

so the total force is $(20-x)0.08+5$. So the work is

$$\int_0^{20}[(20-x)0.08+5]\,dx = \int_0^{20} 16-0.08x+5\,dx = 21x-0.04x^2\Big|_0^{20}$$

$$= 21*20-0.04(20)^2-(0) = 116 \text{ ft-lbs.}$$

② Let x be amount of rope pulled up. Then the force is $66(60-x)$ lbs

work is $\int_0^{60} 66(60-x)\,dx = 66\left[60x-\frac{x^2}{2}\right]_0^{60} = 66\left[\left(60^2-\frac{60^2}{2}\right)-(0)\right]$

$= 66\times\frac{60^2}{2} = 118{,}800$

we want the b s.t. $\int_0^b 66(60-x)\,dx = \frac{18{,}800}{2} = 14{,}400$

$$\int_0^b 66(60-x)\,dx = 66\left(60x-\frac{x^2}{2}\right)\Big|_0^b = 66\left[60b-\frac{b^2}{2}\right]$$

so $66\left[60b-\frac{b^2}{2}\right] = 14{,}400 = 66*60*15$

so $60b-\frac{b^2}{2} = 60*15 = 900$, $\frac{b^2}{2}-60b+900=0$

$b^2-120b+1800=0$

so $b = \frac{120\pm\sqrt{120^2-4*1800}}{2} = 60\pm\frac{\sqrt{7200}}{2} = 60\pm\frac{\sqrt{36*2*100}}{2}$

$= 60\pm\frac{6\times10}{2}\sqrt{2} = 60\pm30\sqrt{2}$

$60+30\sqrt{2} > 60$, so does not make sense. Thus $b = 60-30\sqrt{2} \approx 17.57$ ft

(3) Let x be amount of cable pulled in.

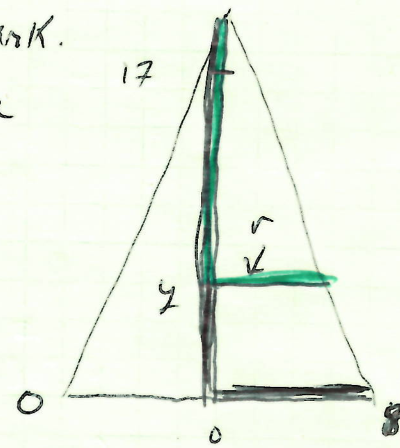
Then when x feet is pulled in, the weight remaining is

$$(500-x)6+1500 = 4500-6x$$

So the work is $\int_0^{500}(4500-6x)\,dx = 4500x-3x^2\Big|_0^{500}$

$$=4500*500-3*500^2 = 1{,}500{,}000 \text{ ft-lbs}$$

(4) The diagram to the right is the cross-section of the tank.



We need the volume of a slice, which is $A\,dy$, where $A=\pi r^2$. We need to compute r in terms of y.

Using similar triangles (dark black and green sides), we can see that

$$\frac{r}{20-y}=\frac{8}{20}, \text{ so } r = 8\,\frac{20-y}{20} = 8-\frac{2}{5}y$$

$$\text{so } A=\pi\left(8-\tfrac{2}{5}y\right)^2=\pi\left(64-\tfrac{32}{5}y+\tfrac{4}{25}y^2\right).$$

We need to pump Dr Pepper from y to a height of 23, so a distance of $23-y$. Thus the work required is

$$\boxed{\int_0^{17}(23-y)\,63\,\pi\left(64-\tfrac{32}{5}y+\tfrac{4}{25}y^2\right)dy}=\pi\int_0^{17}\left(23\cdot 64-\frac{23\cdot 32}{5}y-64y+\frac{32}{5}y^2+\frac{23}{25}4y^2-\frac{4}{25}y^3\right)dy$$

$$=\pi\int_0^{17}\left(1472-\frac{1056}{5}y+\frac{252}{5}y^2-\frac{4}{25}y^3\right)dy=\pi\left[1472y-\frac{528}{5}y^2+\frac{84}{5}y^3-\frac{1}{25}y^4\right]_0^{17}$$

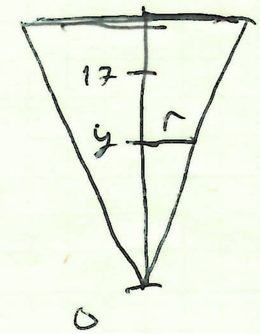
$$=\pi\left[1472*17-\frac{528}{5}*17^2+\frac{84}{5}17^3-\frac{1}{25}17^4\right]=\cancel{[illegible]}\ \frac{3368494}{25}\approx 134739.96$$

should be $8.488\cancel{62}\times 10^6$

Oops! Need to multiply everything by 63 — the force is volume times density

Error somewhere — should be $\boxed{1.5185\times10^6}$

(5) Now $\frac{r}{y}=\frac{8}{20}$, so $r=\frac{2}{5}y$



$$\int_0^{17} 63\pi(23-y)\left(\tfrac{2}{5}y\right)^2 dy = \int_0^{17}\frac{252}{25}\pi(23y^2-y^3)dy$$

$$=\frac{252}{25}\pi\left[\frac{23}{3}y^3-\frac{y^4}{4}\right|_0^{17} = \frac{252}{25}\pi\left[\frac{23}{3}17^3-\frac{17^4}{4}\right]=\frac{4230093}{25}\pi$$

$$\approx 531569.1637$$

definitely less than from #4 which makes sense because in this orientation there is both less total volume and less gets pumped higher.

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(6)

distance, radius

$$\int_0^{11}(18-y)\underbrace{58.9\pi(3.5)^2}_{\text{Force}}dy = \cdots \approx 311676$$

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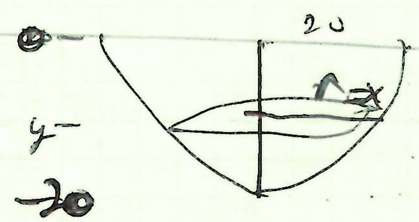
(7) $X=\sqrt{20^2-y^2}$. we will put the tank under the x-axis so the bottom is at -20.



$$\int_{-20}^{-5}(-y)\,62.4\pi\left(\sqrt{20^2-y^2}\right)^2dy=\int_{-20}^{-5}62.4\pi(y^3-400y)\,dy$$

$$=62.4\pi\left[\tfrac{1}{4}y^4-200y^2\right|_{-20}^{-5}=62.4\pi\left[\left(\tfrac{625}{4}-5000\right)-(40{,}000-80{,}000)\right]$$

$$=62.4\pi\left[\tfrac{625}{4}+35{,}000\right]=62.4\pi\,\frac{140625}{4}\approx 6.89107\times10^6 \text{ ft-lbs.}$$

Alternatively, if bottom is 0, integral is

$$\int_0^{15}(20-y)\,62.4\pi\left(\sqrt{40y-y^2}\right)^2dy$$

(8) easy change & add 10 to distance part. so one of these:

$$\int_{-20}^{-5} (10-y)\, 62.4\pi \left(\sqrt{20^2-y^2}\right)^2 dy$$

$$\int_{0}^{15} (30-y)\, 62.4\pi \left(\sqrt{40y-y^2}\right)^2 dy$$

both are about

1.35081×10^7 ft-lbs.

(9)