

Solve the following differential equations. For 1, 3, 4, and 6, your answer should be in the form $y = f(x)$. For 2 and 5 you can leave the equation in another form since solving for y is probably impossible.

1. $\frac{dy}{dx} = \frac{y+1}{x}$

2. $x^2 y^2 dy = (y+1) dx$

3. $(e^{-y} + 1) \sin(x) dx = (1 + \cos(x)) dy, y(0) = 0$

4. $x^2 y' = y - yx, y(-1) = -1$

5. $(y - yx^2) \frac{dy}{dx} = (y+1)^2$

6. $x(x-2)y' + 2y = 0, y(3) = 6$

① solve $\frac{dy}{dx} = \frac{y+1}{x}$

$$\frac{dy}{y+1} = \frac{dx}{x}, \quad \int \frac{1}{y+1} dy = \int \frac{1}{x} dx, \quad \ln|y+1| = \ln|x| + C$$

$$e^{\ln|y+1|} = e^{\ln|x| + C}$$

$$|y+1| = e^{\ln|x|} e^C = |x| \cdot A, \text{ where } A > 0 \text{ (since } e^C > 0 \forall C)$$

Removing absolute values, the sign on the right can be + or -, so we get

$$y+1 = Ax, \text{ where } A \neq 0.$$

$$y = Ax - 1, A \neq 0.$$

Note that if $A=0$, $y=-1$, and $y=-1$ is a solution as well since $y' = 0 = \frac{-1+1}{x}$.

so final solution is $\boxed{y = Ax - 1, A \in \mathbb{R}}$

② solve $x^2 y^2 dy = (y+1) dx$, or $\frac{y^3}{y+1} dy = \frac{1}{x^2} dx$, so $\int \frac{y^3}{y+1} dy = \int \frac{1}{x^2} dx$.

$$\text{Then } \int \frac{y^3}{y+1} dy = \int \frac{(u-1)^2}{u} du = \int \frac{u^2 - 2u + 1}{u} du = \int u - 2 + \frac{1}{u} du = \frac{u^2}{2} - 2u + \ln|u| + C$$

$$\begin{aligned} u &= y+1, \quad y = u-1 \\ du &= dy \end{aligned} \quad = \frac{(y+1)^2}{2} - 2(y+1) + \ln|y+1| + C = \frac{y^2}{2} - y - \frac{3}{2} + \ln|y+1| + C$$

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C', \text{ so we get } \frac{y^2}{2} - y - \frac{3}{2} + \ln|y+1| + C = -\frac{1}{x} + C'$$

We can combine the C, C' , and $-\frac{3}{2}$ into a new C to get

$$\boxed{\frac{y^2}{2} - y + \ln|y+1| = -\frac{1}{x} + C}$$

Which almost certainly can't be solved for y , so we will leave it as is

③ Solve $(e^y + 1) \sin x \, dx = (1 + \cos x) \, dy$, $y(0) = 0$

$$\frac{\sin x}{1 + \cos x} dx = \frac{1}{e^y + 1} dy, \text{ so } \int \frac{\sin x}{1 + \cos x} dx = \int \frac{1}{e^y + 1} dy$$

We'll tackle one at a time:

$$\int \frac{\sin x}{1 + \cos x} dx = -\int \frac{1}{u} du = -\ln|u| = -\ln|1 + \cos x|. \quad \text{[plus } C, \text{ but we can ignore since we'll get one from the other integral]}$$

$u = 1 + \cos x$
 $du = -\sin x \, dx$

$$\int \frac{1}{e^y + 1} dy = \int \frac{1}{\frac{1}{e^y} + 1} dy = \int \frac{1}{\frac{1 + e^y}{e^y}} dy = \int \frac{e^y}{1 + e^y} dy = \ln|1 + e^y| + C$$

$$\text{so } -\ln|1 + \cos x| = \ln|1 + e^y| + C, \text{ or } \ln|1 + e^y| = -\ln|1 + \cos x| + C$$

$$\text{so } e^{\ln|1 + e^y|} = e^{-\ln|1 + \cos x| + C}$$

$$|1 + e^y| = e^{-\ln|1 + \cos x| + C} = A e^{\ln \frac{1}{1 + \cos x}} = \frac{A}{1 + \cos x}, \quad A > 0$$

$$1 + e^y = \pm \frac{A}{1 + \cos x}. \quad A > 0, \text{ so } 1 + e^y = \frac{A}{1 + \cos x}, \quad A \neq 0, \text{ so}$$

$$e^y = \frac{A}{1 + \cos x} - 1, \text{ so } y = \ln\left(\frac{A}{1 + \cos x} - 1\right).$$

$$\text{Finally, } y(0) = 0, \text{ so } 0 = \ln\left(\frac{A}{1 + \cos 0} - 1\right) = \ln\left(\frac{A}{2} - 1\right) = \ln 1$$

$$\text{so } \frac{A}{2} - 1 = 1 \quad (\text{so } \ln\left(\frac{A}{2} - 1\right) = 0), \quad \frac{A}{2} = 2, \quad A = 4,$$

$$\text{Then } \boxed{y = \ln\left(\frac{4}{1 + \cos x} - 1\right)}$$

④ solve $x^2 y' = y - yx$, $y(-1) = -1$.

$$\frac{dy}{dx} = \frac{y(1-x)}{x^2}, \quad \frac{1}{y} dy = \frac{1-x}{x^2} dx, \quad \int \frac{1}{y} dy = \int \frac{1}{x^2} - \frac{1}{x} dx$$

$$\ln|y| = -\frac{1}{x} - \ln|x| + C, \quad e^{\ln|y|} = e^{-\frac{1}{x} - \ln|x| + C}, \text{ so}$$

$$|y| = e^{-\frac{1}{x}} e^{-\ln|x|} e^C = e^{-\frac{1}{x}} e^{\ln \frac{1}{|x|}} A = A e^{-\frac{1}{x}} / |x|, \text{ and}$$

$$\text{so } y = A e^{-\frac{1}{x}} / x, \quad A \neq 0$$

$$y(-1) = -1 \Rightarrow -1 = A e^{-\frac{1}{-1}} / -1 = -Ae, \quad A = e$$

$$\text{so } y = \frac{1}{e} e^{-\frac{1}{x}} / x = \frac{1}{xe^{\frac{1}{x}}} = \frac{1}{xe^{\frac{1}{x}}} \text{ or } \frac{e^{-\frac{1}{x}}}{x}, \text{ i.e. } \boxed{y = \frac{1}{xe^{\frac{1}{x}}}}$$

⑤ solve $(y - yx^2) \frac{dy}{dx} = (y+1)^2$, $y(1-x^2)dy = (y+1)^2 dx$

$$\frac{y}{(y+1)^2} dy = \frac{1}{1-x^2} dx, \text{ so } \int \frac{y}{(y+1)^2} dy = \int \frac{1}{1-x^2} dx$$

$$\text{Left: } \int \frac{y}{(y+1)^2} dy = \int \frac{u-1}{u^2} du = \int \frac{1}{u} - \frac{1}{u^2} du = \ln|u| + \frac{1}{u} + C = \ln|y+1| + \frac{1}{y+1} + C$$

$u = y+1, y = u-1$
 $du = dy$

$$\text{Right: } \int \frac{1}{1-x^2} dx = \int \frac{\cos \theta d\theta}{1 - \sin^2 \theta} = \int \frac{\cos \theta}{\cos^2 \theta} d\theta = \int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C'$$

$x = \sin \theta$
 $dx = \cos \theta d\theta$


 $\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \ln \left| \frac{1+x}{\sqrt{1-x^2}} \right| + C'$

$$\text{so } \boxed{\ln|y+1| + \frac{1}{y+1} = \ln \left| \frac{1+x}{\sqrt{1-x^2}} \right| + C} \quad (\text{where we combined } C' \text{ into } C.)$$

$$(\text{Can be simplified to } \ln|y+1| + \frac{1}{y+1} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C)$$

⑥ Solve $x(x-2)y' + 2y = 0$, $y(3) = 6$

$$x(x-2) \frac{dy}{dx} = -2y, \quad -\frac{1}{2y} dy = \frac{1}{x(x-2)} dx, \quad \int -\frac{1}{2y} dy = \int \frac{1}{x(x-2)} dx$$

LHS: $\int -\frac{1}{2y} dy = -\frac{1}{2} \int \frac{1}{y} dy = -\frac{1}{2} \ln|y|$. (Ignoring constant)

RHS: $\int \frac{1}{x(x-2)} dx = \int \frac{-1}{2x} + \frac{1}{2(x-2)} dx$ (using partial fraction decomposition - steps not shown.)

$$= -\frac{1}{2} \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{x-2} dx = -\frac{1}{2} \ln|x| + \frac{1}{2} \ln|x-2| + C$$

$$= -\frac{1}{2} (\ln|x| - \ln|x-2|) + C = -\frac{1}{2} \ln\left|\frac{x}{x-2}\right| + C$$

so $-\frac{1}{2} \ln|y| = -\frac{1}{2} \ln\left|\frac{x}{x-2}\right| + C$, so $\ln|y| = \ln\left|\frac{x}{x-2}\right| + C'$ ($C' = -2C$)

$$\begin{aligned} \ln|y| &= \ln\left|\frac{x}{x-2}\right| + C' = e^{\ln\left|\frac{x}{x-2}\right|} e^{C'} = A \left|\frac{x}{x-2}\right|, A > 0 \\ \text{ii} \\ |y| & \end{aligned}$$

so $|y| = A \left|\frac{x}{x-2}\right|$, $A > 0$, or $y = A \frac{x}{x-2}$ $A \neq 0$,

$y(3) = 6 \Rightarrow 6 = A \frac{3}{3-2} = 3A$, $A = 2$. so

$$\boxed{y = 2 \frac{x}{x-2}}$$