Example 1 (Exercise 4 from the textbook):

A feature of radioactive decay is that the amount of a radioactive substance decreases at a rate proportional to the current amount of the substance. The *half life* of a substance is the amount of time it takes for half of a given amount of substance to decay. The half life of carbon-14 is approximately 5730 years. If an ancient object has a carbon-14 amount that is 20% of the original amount, how old is the object?

Let C(t) be the amount of carbon-14 in the object. The differential equation in this case is dC/dt = kC.

Then $\frac{1}{C}dC = k\,dt$, $\int \frac{1}{C}dC = \int k\,dt$, $\ln|C| = kt + d$,

So $e^{\ln|C|} = e^{kt+d} = e^{kt}e^{d} = Ae^{kt}$, where $A > 0$

$|C| = Ae^{kt}$, $A > 0$

$C = \pm Ae^{kt}$, $A > 0$

$C = Ae^{kt}$, $A \neq 0$. if $t=0$, $C = A$, so A is the initial amount.

we know $Ae^{k5730} = \frac{1}{2}A$, $e^{k\cdot 5730} = \frac{1}{2}$ $k\cdot 5730 = \ln\frac{1}{2}$

$k = \frac{\ln 1/2}{5730} = \frac{-\ln 2}{5730}$.

So $C = Ae^{\frac{-\ln 2\, t}{5730}}$

$.2A = Ae^{\frac{-\ln 2\, t}{5730}}$, $\frac{-\ln 2\, t}{5730} = \ln .2$; $t = \frac{-\ln .2}{\ln 2}5730 = 13304.15$ years old.

Example 2: Newton's Law of Cooling is the simple assumption that the temperature of an object changes at a rate proportional to the difference between the temperature of the object and the ambient temperature of the room. If T is the temperature of the object and A is the constant ambient temperature, Newton's Law of Cooling can be expressed as the differential equation

$$dT/dt = k(A-T).$$

So, I put a bottle of Dr. Pepper in the freezer to cool it down. It was in my car, so it is pretty warm at 32 C (about 90 degrees Fahrenheit). My freezer is -20 C. After 10 minutes, my Dr. Pepper has cooled to about 27 C. How long will it take to cool to 1 C (shortly before it freezes)?

we have $\frac{dT}{dt} = k(-20-T)$, so $\frac{-1}{20+T}\,dT = k\,dt$, $\int -\frac{1}{20+T}\,dT = \int k\,dt$

$-\ln|20+T| = kt + C$, for simplicity, $\ln|20+T| = -kt - C$

$e^{\ln|20+T|} = e^{-kt}e^{-C} = e^{-kT}A$, $A>0$

$|20+T| = Ae^{-kT}$, $A>0$

$20+T = Ae^{-kt}$, $A \neq 0$

$T = Ae^{-kt} - 20$, $A \neq 0$, $T(0) = 32 = Ae^{0} - 20$, so $A = 52$

$T(10) = 27$, so $\cancel{27 = Ae^{-k10}}$ $27 = 52e^{-k10} - 20$ $e^{-k10} = \frac{47}{52}$

$-k10 = \ln\frac{47}{52}$, $k = -\frac{1}{10}\ln\frac{47}{52}$

$T = 52e^{+\frac{1}{10}\ln\frac{47}{52}t} - 20$

we want $1 = 52e^{\frac{1}{10}\ln\frac{47}{52}t} - 20$, $\frac{21}{52} = e^{\frac{1}{10}\ln\frac{47}{52}t}$ $\ln\frac{21}{52} = \frac{1}{10}\ln\frac{47}{52}t$

$t = 10\ln(\frac{21}{52})/\ln(\frac{47}{52}) \approx 89.689$ minutes

Example 3 (based on Example 8.4.7): Suppose a disease spreads through a population at a rate proportional to the product of the number of infected and uninfected individuals. If 4% of the population is sick initially and 12% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t.

The differential equation in this case is dy/dt = k y(1-y).

we have $\frac{dy}{dt} = k\,y(1-y) \Rightarrow \frac{1}{y(1-y)}\,dy = k\,dt \Rightarrow \int \frac{1}{y(1-y)}\,dy = \int k\,dt$

minus.

$\int \frac{1}{y(1-y)}\,dy = \int \frac{1}{y} + \frac{1}{1-y}\,dy = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|$

so $\ln\left|\frac{y}{1-y}\right| = kt + C$, $\frac{y}{1-y} = e^{kt+C} = Ae^{kt}$, $A > 0$

$\int k\,dt = kt + C$,

Then $y = (1-y)Ae^{kt} = Ae^{kt} - yAe^{kt}$, so $y + yAe^{kt} = Ae^{kt}$

$y(1 + Ae^{kt}) = Ae^{kt}$, $y = \frac{Ae^{kt}}{1+Ae^{kt}} \cdot \frac{\frac{1}{Ae^{kt}}}{\frac{1}{Ae^{kt}}} = \frac{1}{\frac{1}{Ae^{kt}}+1} = \frac{1}{be^{-kt}+1}$, where $b = \frac{1}{A}$.

$y(0) = .04 = \frac{1}{be^{-k0}+1} = \frac{1}{b+1}$, $b+1 = \frac{1}{.04}$ $b = 24$

so $y = \frac{1}{24e^{-kt}+1}$

$y(7) = .12$, so $.12 = \frac{1}{24e^{-k7}+1}$, $24e^{-k7}+1 = \frac{1}{.12} = \frac{100}{12}$

so $24e^{-k7} = \frac{88}{12}$, $e^{-k7} = \frac{88}{12 \cdot 24}$, $-k7 = \ln\left|\frac{88}{12 \cdot 24}\right| = \ln\frac{11}{36}$

$k = -\frac{1}{7}\ln\frac{11}{36}$, so

$y = \frac{1}{24e^{\frac{1}{7}\ln\frac{11}{36}t}+1}$

e.g. 2 weeks: $y = \frac{1}{24e^{\frac{1}{7}\ln\frac{11}{36}\cdot 14}+1} \approx .3086$