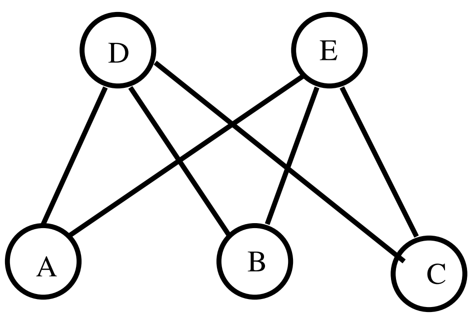
Graphs

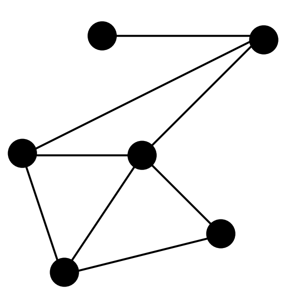
- A **(simple) graph** $G = (V, E)$ consists of
  - V, a nonempty set of *vertices* and
  - E, a set of *unordered* pairs of distinct vertices called *edges*.

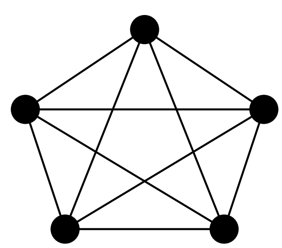
Examples

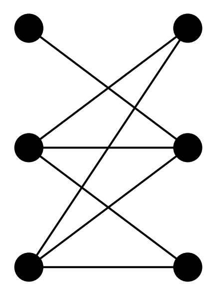


V={A,B,C,D,E}

E={ (A,D),(A,E),(B,D),
(B,E),(C,D),(C,E)}

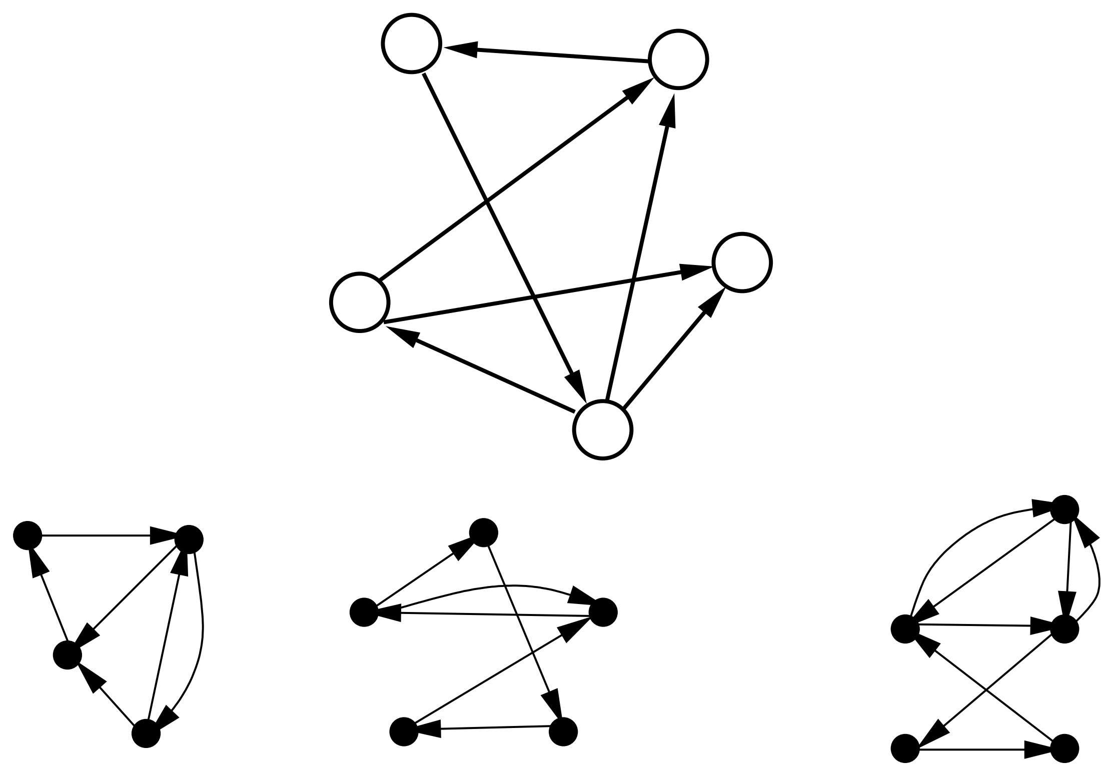






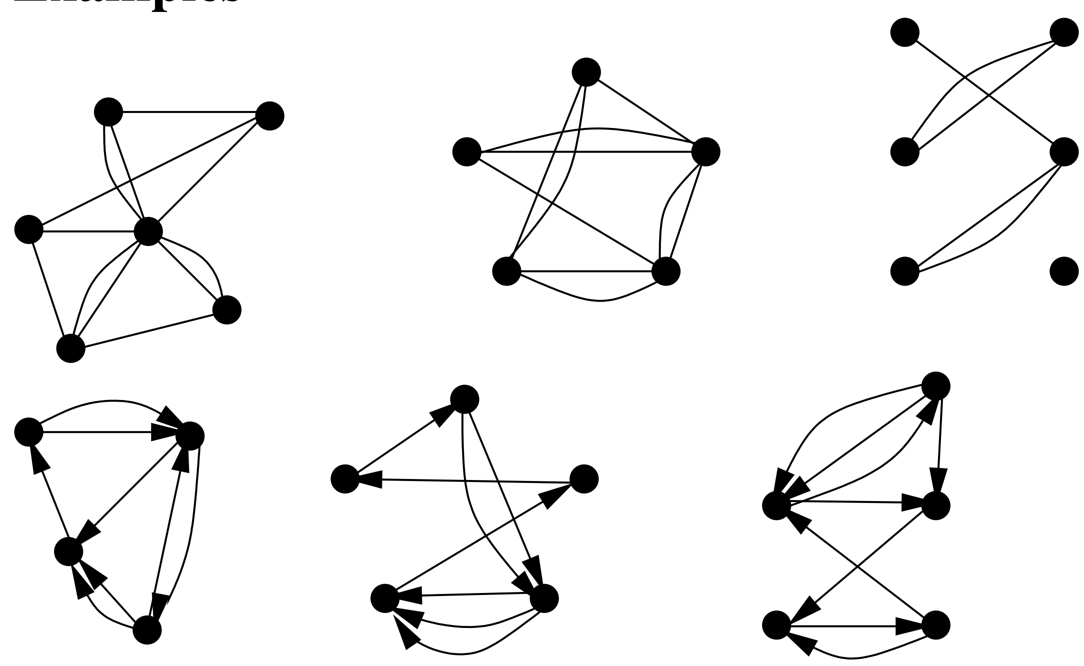
Directed Graphs

- A **directed graph** (or **digraph**) $G = (V, E)$ consists of
  - V, a nonempty set of *vertices* and
  - E, a set of *ordered* pairs of distinct vertices called *edges*.
- **Examples**



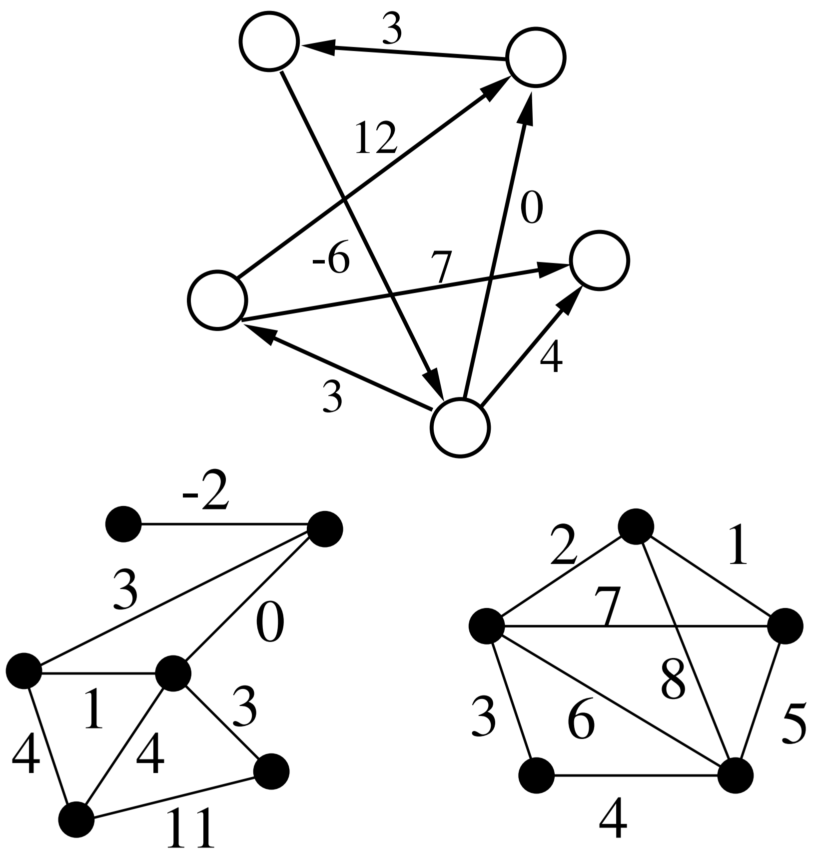
Multigraphs

- A **multigraph (directed multigraph)** $G = (V, E)$ consists of
  - V, a set of vertices,
  - E, a set of edges, and
  - a function f from E to $\{\{u, v\} : u \neq v \in V\}$ (function f from E to $\{(u, v) : u \neq v \in V\}$.)
- Two edges e_1 and e_2 with $f(e_1) = f(e_2)$ are called *multiple edges*.
- Put simply, a **multigraph** $G = (V, E)$ is a graph in which multiple edges are allowed.
- **Examples**



Weighted Graphs

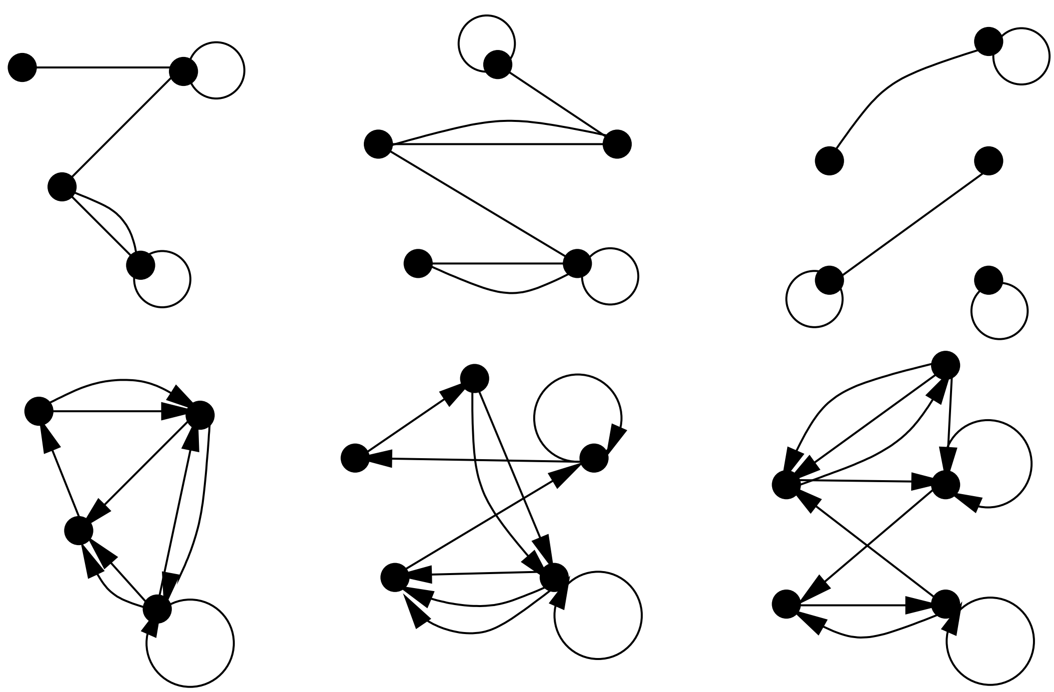
- A **weighted graph** is a graph (or digraph) with the additional property that each edge e has associated with it a real number $w(e)$ called it's *weight*.
- A weighted digraph is often called a **network**.
- **Examples**



Psuedographs

- A **psuedograph** $G = (V, E)$ consists of
  - V, a set of vertices,
  - E a set of edges, and
  - a function f from E to $\{\{u, v\} : u, v \in V\}$.
- **Psuedo-multigraphs** are defined similarly.
- Put another way, a *psuedograph* is a graph in which we allow *loops*, that is, edges from a vertex to itself.

Examples



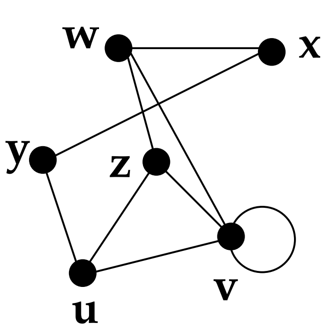
Graph Definitions Summary

- There are several ways to categorize graphs:
  - Directed or undirected edges.
  - Weighted or unweighted edges.
  - Allow multiple edges or not.
  - Allow loops or not.
- Unless specified, you can usually assume a graph does not allow multiple edges and loops. These aren't that common.
- For clarity, if a graph is not specified as weighted or directed, assume it isn't.
- The most common graphs we'll use are graphs, digraphs, weighted graphs, and networks.
- When writing graph algorithms, it is important to know what characteristics the graphs have. For instance, if a graph might have loops, the algorithm should be able to handle it.

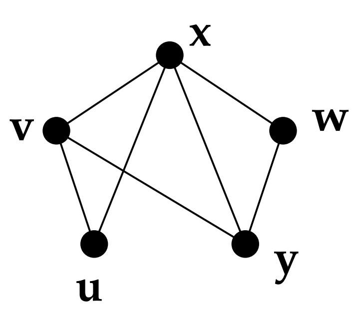
Graph Terminology

Let u and v be vertices, and let $e = \{u, v\}$ be an edge in an undirected graph G.

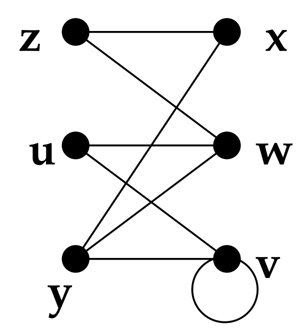
- The vertices u and v are said to be **adjacent**
- The edge e is said to **incident with** u and v.
- The edge e is said to **connect** u and v.
- The vertices u and v are called the **endpoints** of the edge e.
- The **degree** of a vertex, denoted $deg(v)$, in an undirected graph is the number of edges incident with it (where loops are counted twice).



G_1

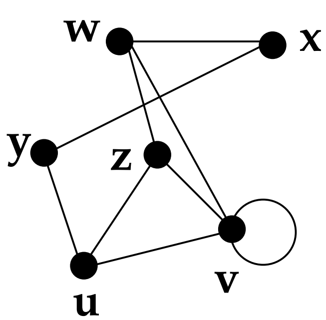


G_2

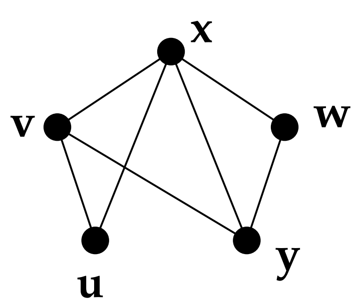


G_3

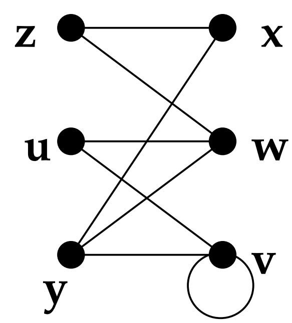
Examples



G_1



G_2

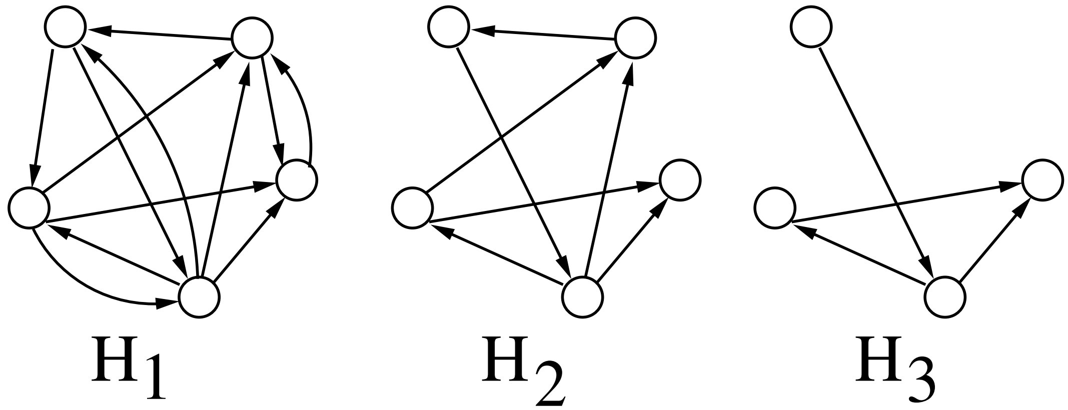


G_3

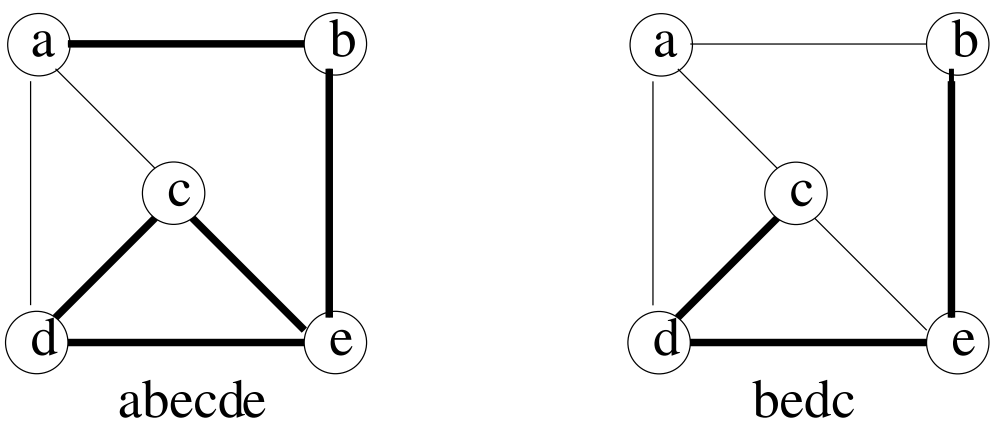
G_1	G_2	G_3
deg(u)=3	deg(u)=2	deg(u)=2
deg(v)=5	deg(v)=3	deg(v)=4
deg(w)=3	deg(w)=2	deg(w)=3
deg(x)=2	deg(x)=4	deg(x)=2
deg(y)=2	deg(y)=3	deg(y)=3
deg(z)=3		deg(z)=2

More Graph Terminology

- A **subgraph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

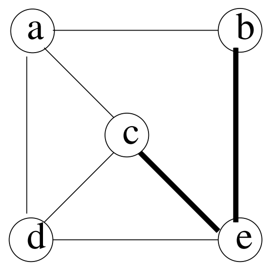


- A **path** is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that consecutive vertices v_i and v_{i+1} are adjacent



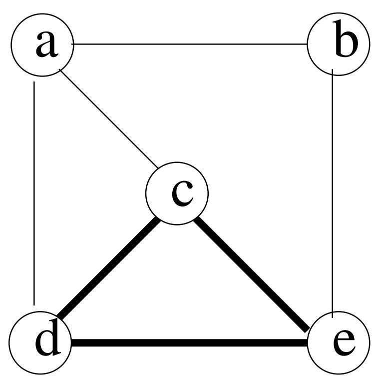
More Graph Terminology

- A **simple path** is a path with no repeated vertices.



bec

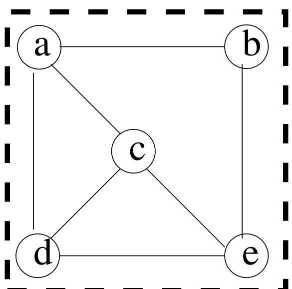
- A **cycle** is a simple path whose last vertex is the same as the first vertex.



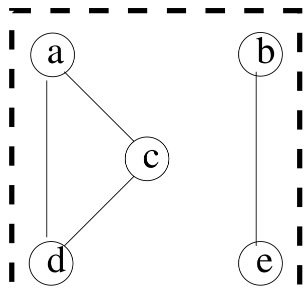
cdec

More Graph Terminology

- A graph is called **connected** if there is a path between every pair of distinct vertices.

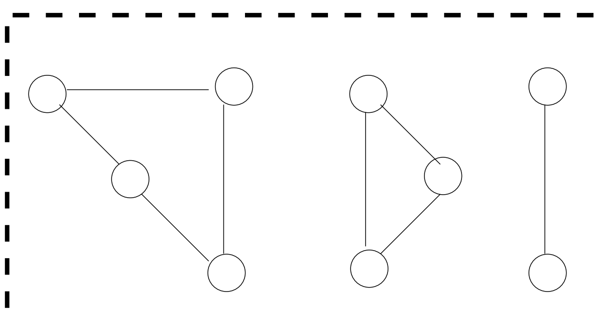


connected



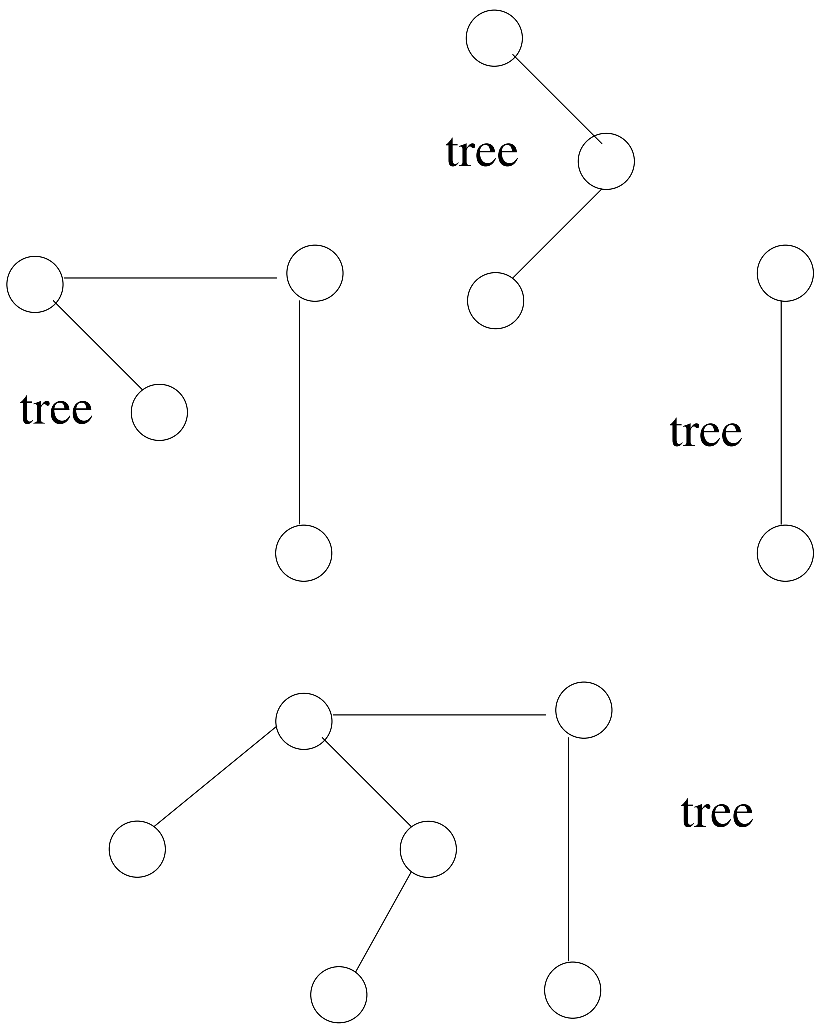
not connected

- A **connected component** of a graph is a maximal connected subgraph. e.g. the graph below has 3 connected components.



Trees

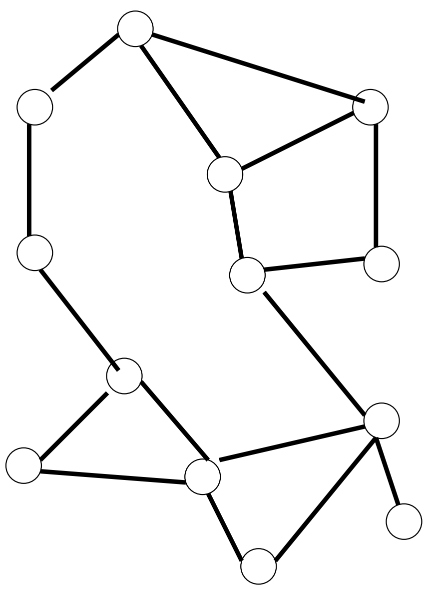
- A **tree** (or **unrooted tree**) is a connected acyclic graph. That is, a graph with no cycles.
- A **forest** is a collection of trees.



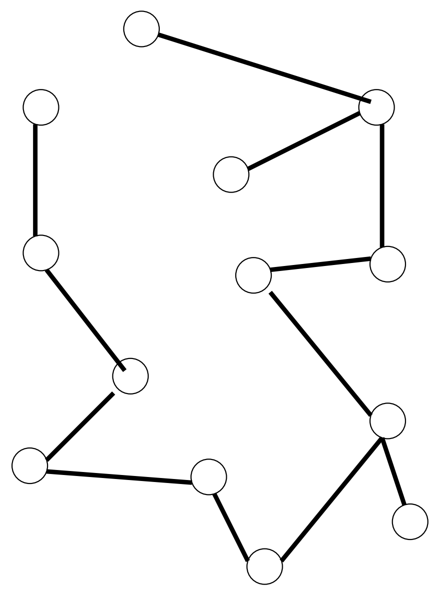
- These trees are not to be confused with *rooted trees*, which we will see later.

Spanning Tree

- A **spanning tree** of G is a subgraph which is a tree and contains all of the vertices of G.



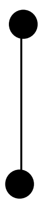
G



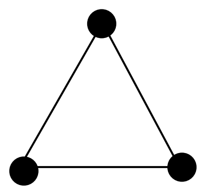
spanning tree of G

Some Special Graphs

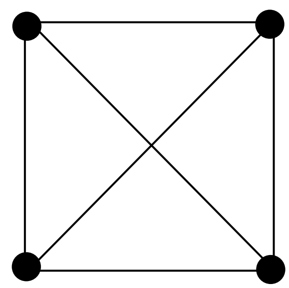
- K_n: The **complete graph** on n vertices.



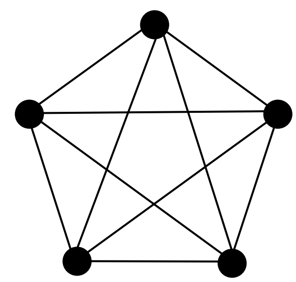
$\mathbf{K_2}$



$\mathbf{K_3}$

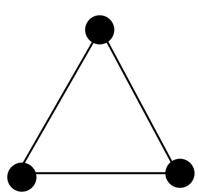


$\mathbf{K_4}$

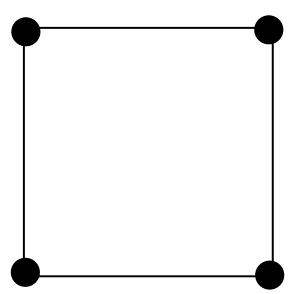


$\mathbf{K_5}$

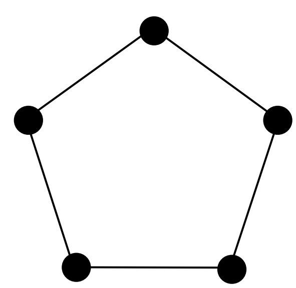
- C_n: The **cycle** of length n.



$\mathbf{C_3}$



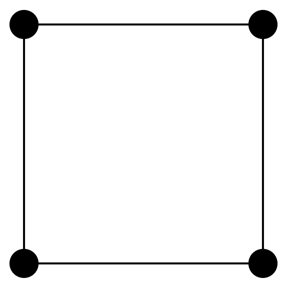
$\mathbf{C_4}$



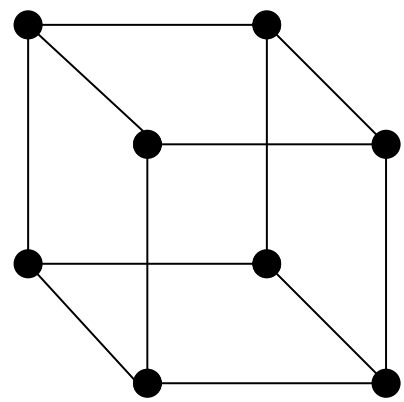
$\mathbf{C_5}$

- Q_n: The n-**cube**.

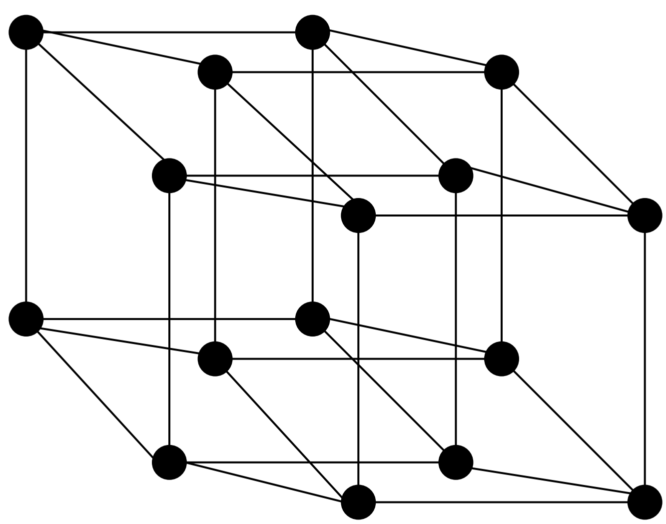
$\mathbf{Q_1}$



$\mathbf{Q_2}$

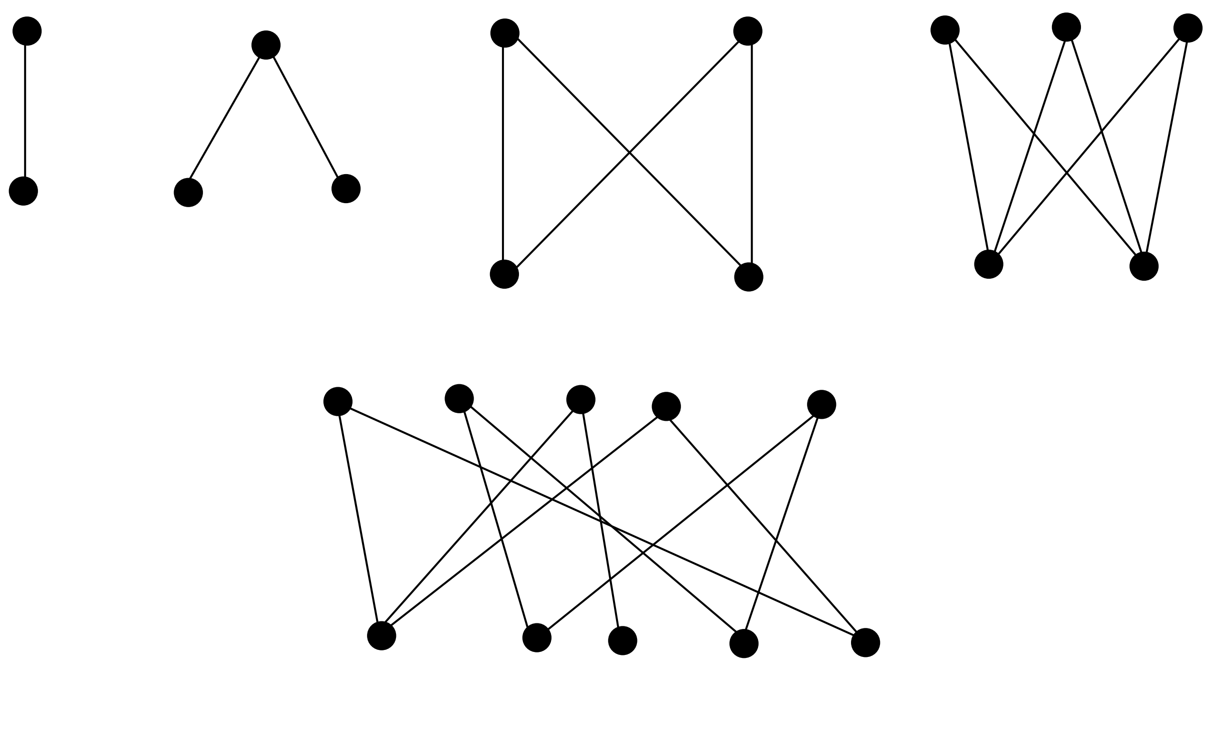


$\mathbf{Q_3}$



$\mathbf{Q_4}$

- **Bipartite Graphs:** A simple graph G is called **bipartite** if the vertex set V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge connects a vertex in V_1 to a vertex in V_2.
  Put another way, no edges in V_1 are connected to each other, and no edges in V_2 are connected to each other.



Some Theorems

- **Theorem 1:** Let $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} deg(v).$$

- **Proof:** Let $X = \{(e, v) : e \in E, v \in V, e \text{ and } v \text{ are incident}\}$. We will compute $|X|$ in two ways. Each edge $e \in E$, is incident with exactly 2 vertices. Thus,

$$|X| = 2e.$$

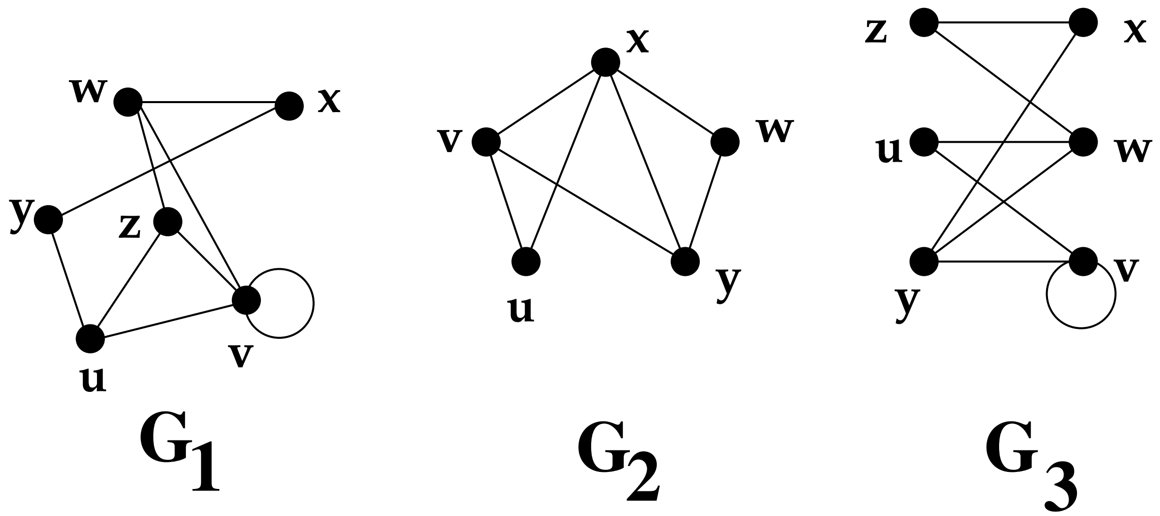
  Also, each vertex $v \in V$ is incident with deg(v) edges. Thus, we have that

$$|X| = \sum_{v \in V} deg(v).$$

  Setting these equal, we have the result ■

- **Corollary 2:** An undirected graph has an even number of vertices of odd degree.

Examples



Graph	G_1	G_2	G_3
$\lvert E\rvert$	9	7	8
$\sum_{v\in V} deg(v)$	18	14	16

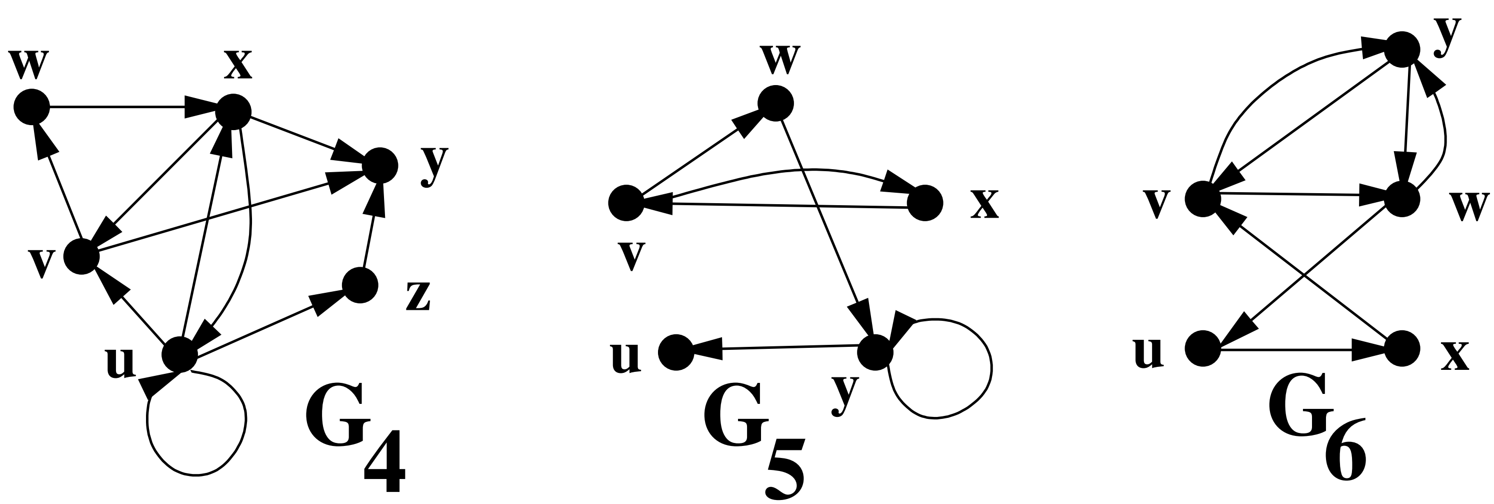
Directed Graph Terminology

Let u and v be vertices in a directed graph G, and let $e = (u, v)$ be an edge in G.

- u is said to be **adjacent to** v.
- v is said to be **adjacent from** u.
- u is called the **initial vertex** of (u, v).
- v is called the **terminal** or **end vertex** of (u, v).
- The **in-degree** of u, denoted by $deg^-(u)$, is the number of edges in G which have u as their terminal vertex.
- The **out-degree** of u, denoted by $deg^+(u)$, is the number of edges in G which have u as their initial vertex.
- **Theorem 3:** Let $G = (V, E)$ be a directed graph. Then

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|.$$

Examples



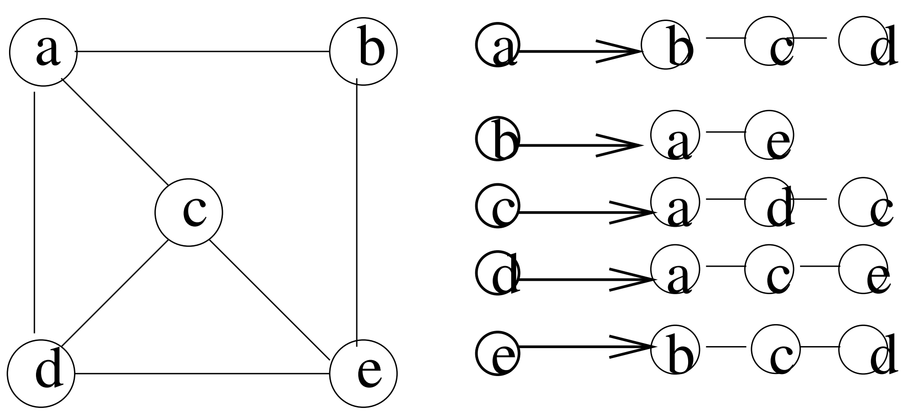
G_4		G_5		G_6	
$\deg^-(u)=2$	$\deg^+(u)=4$	$\deg^-(u)=1$	$\deg^+(u)=0$	$\deg^-(u)=1$	$\deg^+(u)=1$
$\deg^-(v)=2$	$\deg^+(v)=2$	$\deg^-(v)=1$	$\deg^+(v)=2$	$\deg^-(v)=2$	$\deg^+(v)=2$
$\deg^-(w)=1$	$\deg^+(w)=1$	$\deg^-(w)=1$	$\deg^+(w)=1$	$\deg^-(w)=2$	$\deg^+(w)=2$
$\deg^-(x)=2$	$\deg^+(x)=3$	$\deg^-(x)=1$	$\deg^+(x)=1$	$\deg^-(x)=1$	$\deg^+(x)=1$
$\deg^-(y)=3$	$\deg^+(y)=0$	$\deg^-(y)=2$	$\deg^+(y)=2$	$\deg^-(y)=2$	$\deg^+(y)=2$
$\deg^-(z)=1$	$\deg^+(z)=1$				

Graph Representation

There are two common ways of representing G. Let $G = (V, E)$ be a graph with n vertices and m edges.

Adjacency Lists

- For each vertex v in G, we store a list of vertices adjacent to v.
- This is often implemented using linked lists.



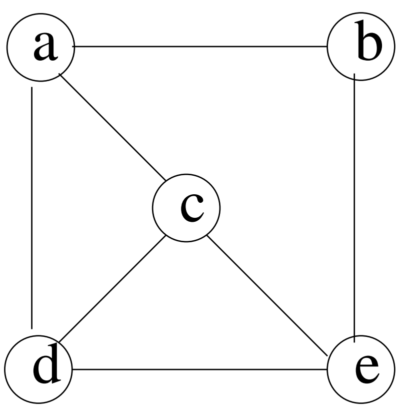
- For weighted graphs, an additional field can be stored in each node.
- The space required for storage is $\Theta(n + 2m) = \Theta(n + m)$ for graphs, and $\Theta(n + m) = \Theta(n + m)$ for digraphs.

Adjacency Matrix

- Number the vertices $1, 2, \ldots, n$ in some arbitrary order.
- We use a n by n matrix M defined as

$$M(i,j) = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge} \\ 0 & \text{if } (i,j) \text{ is not an edge} \end{cases}$$

- If G is weighted, we store the weights in the matrix. For non-adjacent vertices, we store ∞, or *MAX_INT*.
- It is clear that this representation requires $\Theta(n^2)$ space.



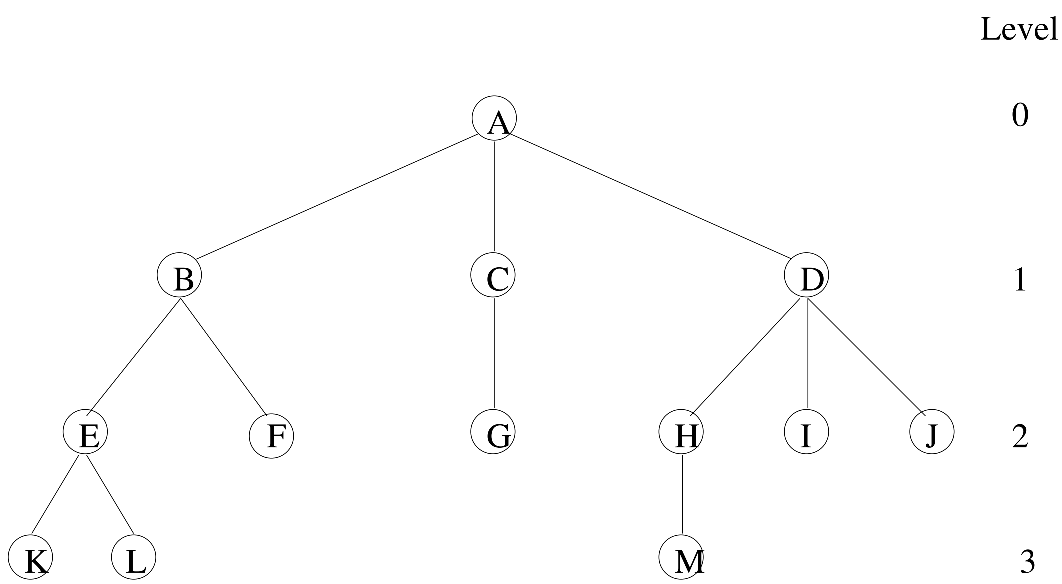
	a	b	c	d	e
a	0	1	1	1	0
b	1	0	0	0	1
c	1	0	0	1	1
d	1	0	1	0	1
e	0	1	1	1	0

Some Basic Graph Problems

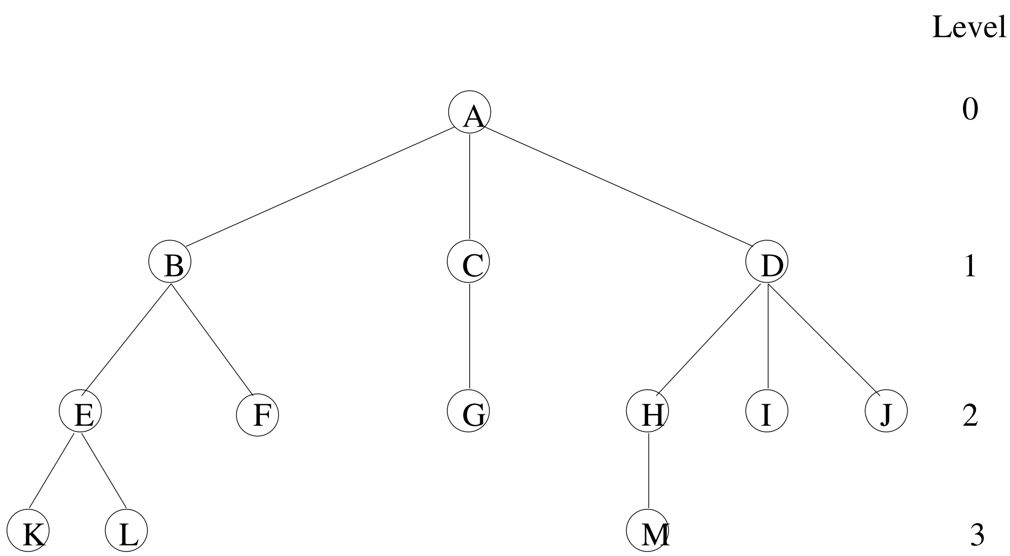
- Is there a path from A to B?
- CYCLES: Does the graph contain a cycle?
- CONNECTIVITY (SPANNING TREE): Is there a way connect the vertices?
- BICONNECTIVITY: Will the graph become disconnected if one vertex is removed?
- PLANARITY: Is there a way to draw the graph without edges crossing?
- SHORTEST PATH: What is the shortest way from A to B?
- LONGEST PATH: What is the longest way from A to B?
- MINIMAL SPANNING TREE: What is the best way connect the vertices?
- TRAVELING SALESMAN: What is the shortest route to connect the vertices without visiting the same vertex twice?

(Rooted) Trees

- A **rooted tree** is a tree which has a specially designated vertex called the *root*.
- In rooted trees, vertices are called *nodes*.
- Each node contains some information and one or more links to other nodes further down the hierarchy. (Similar to nodes in a linked list.)
- For convenience, we can think of trees as acyclic digraphs in which every edge "points away from" the root.

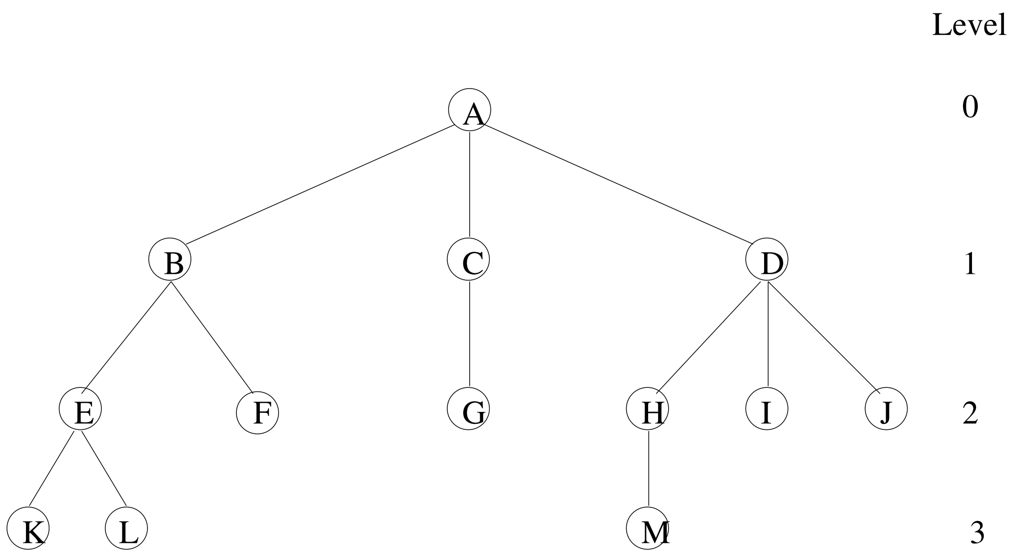


Rooted Trees Terminology



- A node that is *adjacent from* v is a **child** of v.
- A node that is *adjacent to* v is a **parent** of v.
- Two nodes who have the same parent are **siblings**.
- A **leaf** or **external node** is a node with degree zero. (i.e. a node with no children)
- An **internal node** is a nonleaf node. (i.e. a node with at least one child)
- An **ancestor** of a node is any node on the path from the root to the node.
- A **descendant** of a node v is any node which has v as an ancestor.

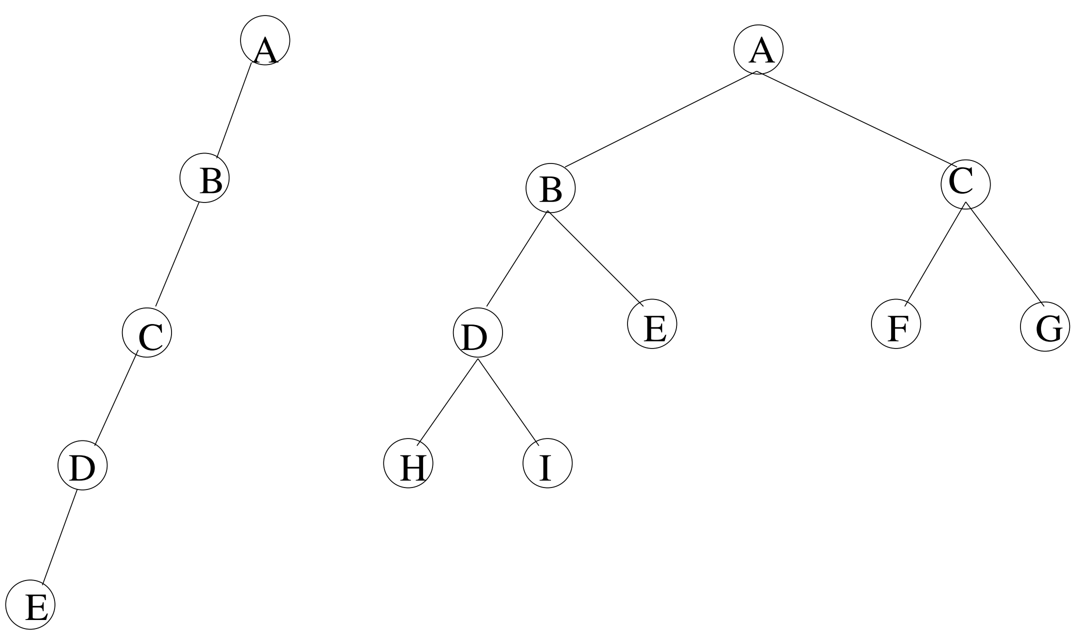
Rooted Trees Terminology



- The **degree of a node** is the number of children the node has.
- The **depth** of a node is the length of the path from the root to the node.
- The **height** of a node is the maximum length of a path from the node to a leaf.
- The **height or depth** of a tree is the maximum height of any node in the tree.
- The **subtree rooted at** x is the subtree consisting of x and all of its descendents.

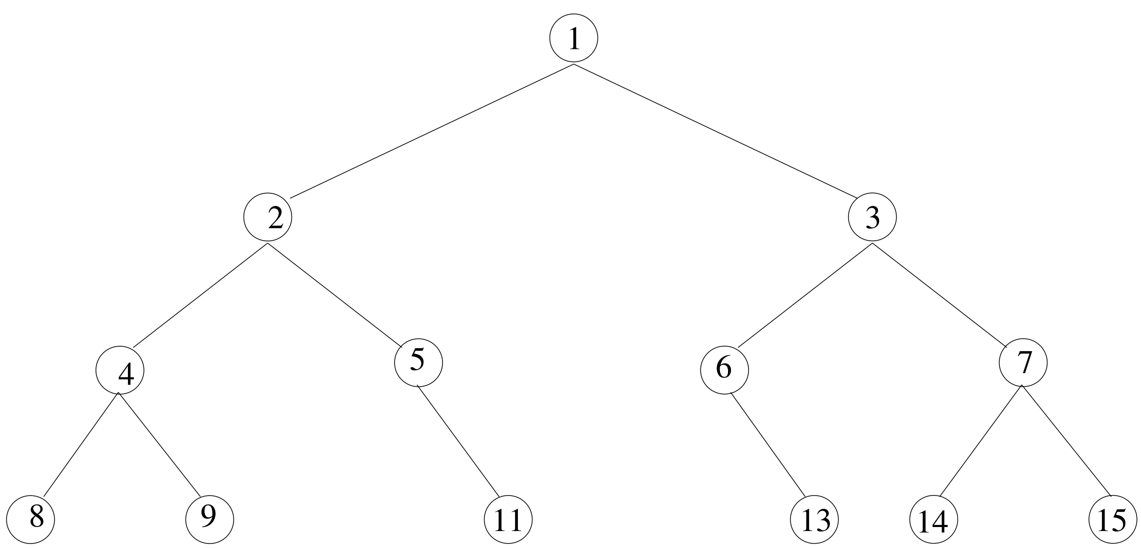
Binary Tree

- A **binary tree** is a finite set of nodes that is either empty or consists of a root and two disjoint binary trees called the *left subtree* and the *right subtree*
- Put another way, a **binary tree** is a rooted tree such that each node has
  - no children,
  - a *right child*,
  - a *left child*, or
  - both a *right child* and a *left child*.



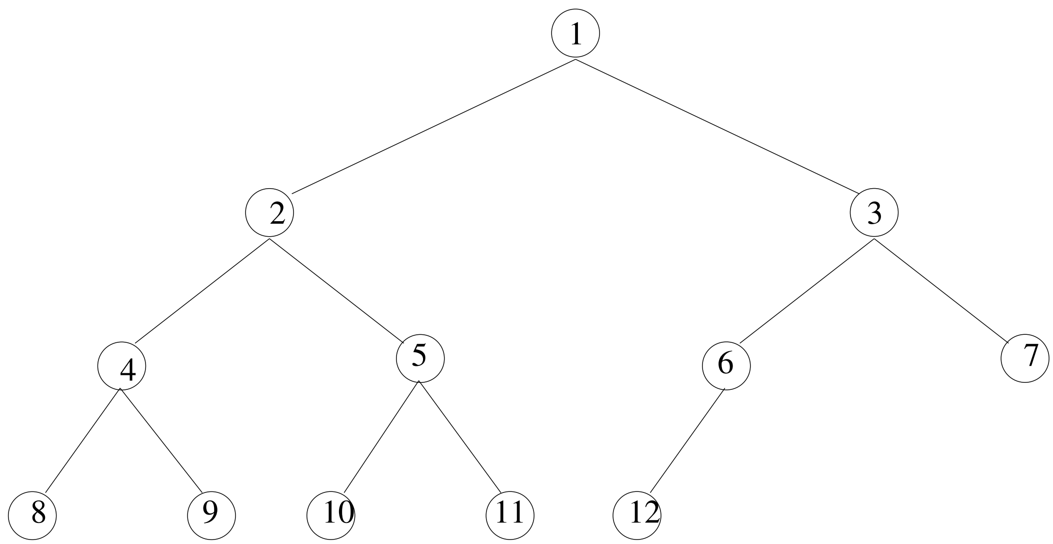
Binary Trees: Definitions

- A **full binary tree** is one in which internal nodes completely fill every level, except possibly the last.
- A **complete binary tree** is a full binary tree where the internal nodes on the bottom level all appear to the left of the external nodes on that level.
- **Example:** A full binary tree

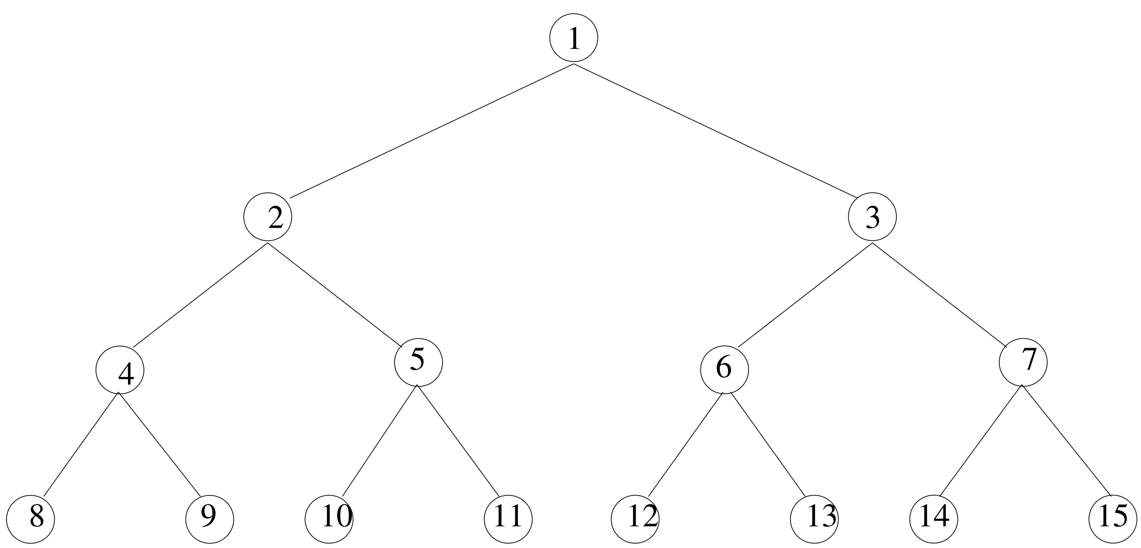


Binary Tree Examples

- **Example:** A complete binary tree

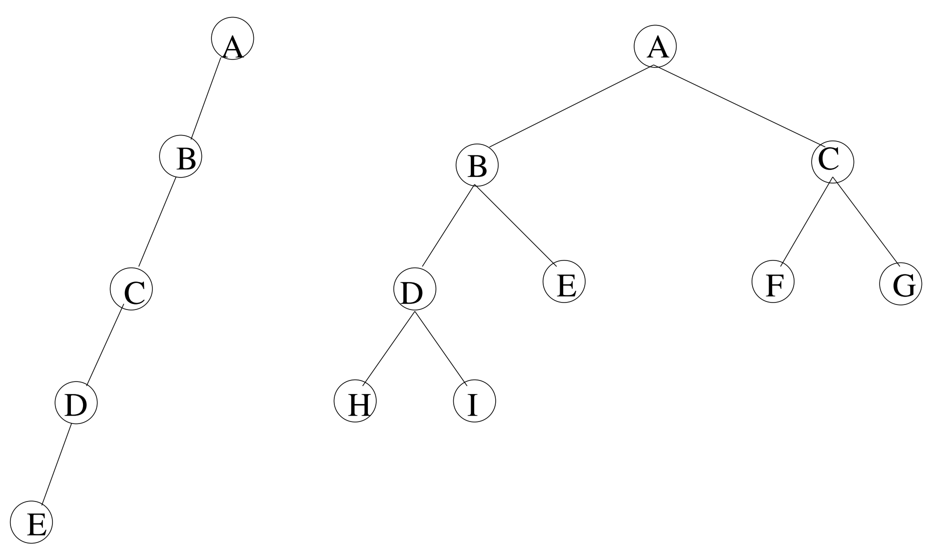


- **Example:** A totally complete binary tree



Properties of Binary Trees

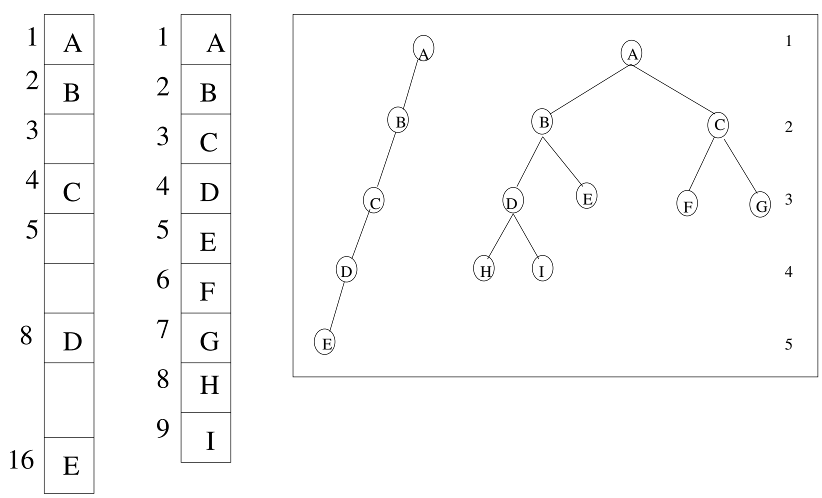
- The maximum number of nodes on level i of a binary tree is $2^i, i \geq 1$.
- The maximum number of nodes in a binary tree of depth k is $2^{k+1} - 1, k \geq 1$.
- There is exactly one path connecting any two nodes in a tree
- A tree with n nodes has $n - 1$ edges
- The height of a full binary tree with n internal nodes is about $\log_2 n$.



Binary Tree Representation: Arrays

Let G be a tree of height $\log_2 n$ with m nodes, where $m \leq n$. We can represent G with an array A of size n. $A[1]$ is the root, and given a node with index i, we can find the index of parents and children as follows:

- $parent(i) = \begin{cases} \lfloor i/2 \rfloor & \text{if } i \neq 1 \\ \text{undefined} & \text{if } i = 1 \end{cases}$
- $left(i) = \begin{cases} 2i & \text{if } 2i \leq n \\ \text{undefined} & \text{if } 2i > n \end{cases}$
- $right(i) = \begin{cases} 2i+1 & \text{if } 2i+1 \leq n \\ \text{undefined} & \text{if } 2i+1 > n \end{cases}$

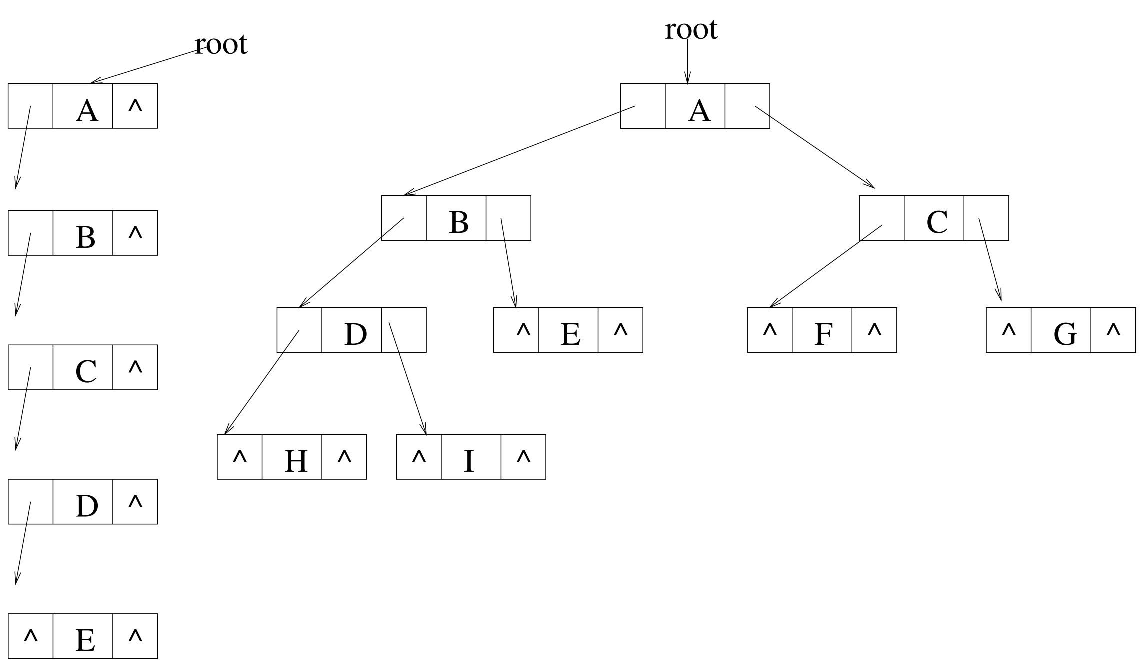


Binary Tree Representation: Linked Lists

- We define a tree node by

```
struct treenode{
   int data;
   treenode *left_child;
   treenode *right_child;
   };
```

- We find children by following the pointers.
- Parents are harder to find, unless we use doubly linked lists.



Binary Tree Traversals

- When we visit each node in the tree exactly once, we say we have **Traversed** the tree.
- A full traversal produces a linear order of the information in a tree.
- There are several ways to traverse a tree.
  - **preorder**: visit a node, then traverse its left subtree, and then traverse its right subtree.
  - **inorder**: traverse the left subtree, visit the node and then traverse its right subtree
  - **postorder**: first traverse the left subtree, traverse the right subtree, and then visit the node.
- There is a natural correspondence between these traversals and producing the prefix, infix and postfix form of an expression.

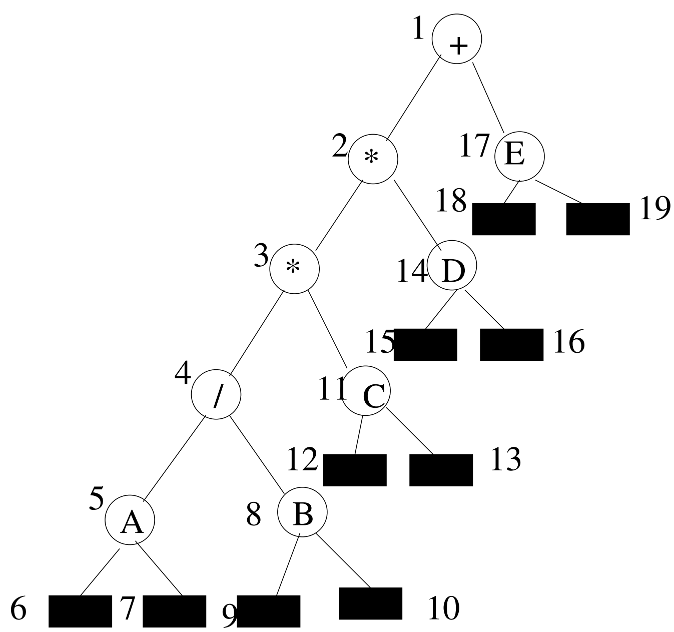
Tree Traversal

- **preorder**: visit a node, then traverse its left subtree, and then traverse its right subtree.
- **inorder**: traverse the left subtree, visit the node and then traverse its right subtree.
- **postorder**: first traverse the left subtree, traverse the right subtree.

Preorder: + * * / A B C D E

Inorder:A / B * C * D + E
nfix form of the expression

Postorder: A B / C * D * E +



Tree Traversal: Implementation

- Assume we have used a linked list to implement a tree.

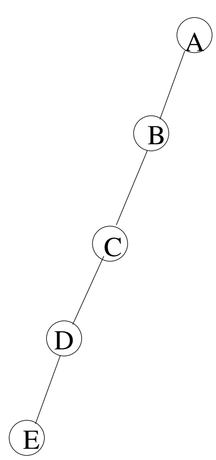
```
struct tree_node{
   int data;
   tree_node *left_child;
   tree_node *right_child;
   };
```

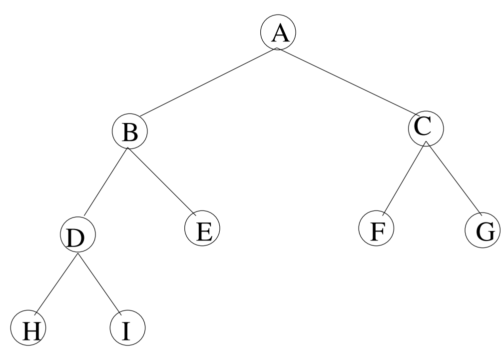
- Assume we have a pointer to the root node.
- From this, we can traverse the tree with any of the methods.

Preorder Traversal

Visit a node, then traverse its left subtree, and then traverse its right subtree.

```
void PreOrderTraversal(treenode *ptr) {
    if (ptr!=0) {
          cout << ptr->data;
          PreOrderTraversal(ptr->left);
          PreOrderTraversal(ptr->right);
          }
    }
```

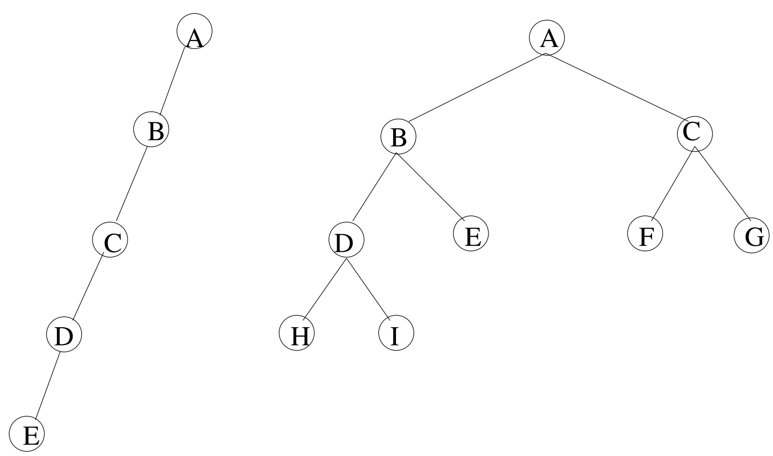




Postorder Traversal

First traverse the left subtree, traverse the right subtree, and finally visit the node.

```
void PostOrderTraversal(treenode *ptr) {
    if (ptr!=0) {
         PostOrderTraversal(ptr->left);
         PostOrderTraversal(ptr->right);
         cout << ptr->data;
         }
    }
```



Inorder Traversal

Traverse the left subtree, visit the node and then traverse its right subtree.

```
void InOrderTraversal(treenode *ptr) {
    if (ptr!=0) {
         InOrderTraversal(ptr->left);
         cout << ptr->data;
         InOrderTraversal(ptr->right);
         }
    }
```