Searching Lists

- There are many instanced when one is interested in storing and searching a list:
  - A phone company wants to provide caller ID: Given a phone number a name is returned.
  - Somebody who plays the *Lotto* thinks they can increase their chance to win if they keep track of the winning number from the last 3 years.
- Both of these have simple solutions using arrays, linked lists, binary trees, etc.
- Unfortunately, none of these solutions is adequate, as we will see.

Problematic List Searching

Here are some solutions to the problems:

- For caller ID:
  - Use an array indexed by phone number.
    - Requires one operation to return name.
    - An array of size 1,000,000,000 is needed.
  - Use a linked list.
    - This requires $O(n)$ time and space, where n is the number of *actual* phone numbers.
    - This is the best we can do space-wise, but the time required is horrendous.
  - A balanced binary tree: $O(n)$ space and $O(\log n)$ time. Better, but not good enough.
- For the **Lotto** we could use the same solutions.
  - The number of possible lotto number is 15,890,700 for a 6/50 lotto.
  - The number of entries stored is on the order of 1000, assuming a daily lotto.

The Hash Table Solution

- A **Hash table** is similar to an array:
  - Has fixed size m.
  - An item with key k goes into index $h(k)$, where h is a function from the keyspace to $\{0, 1, \ldots, m-1\}$
- The function h is called a **hash function**.
- We call $h(k)$ the **hash value** of k.
- A **hash table** allows us to store values from a large set in a small array in such a way that searching is fast.
- The space required to store n numbers in a hash table of size m is $O(m+n)$. *Notice that this does not depend on the size of the keyspace.*
- The average time required to insert, delete, and search for an element in a hash table is $O(1)$.
- Sounds perfect. Then what's the problem? Let's look at an example.

Hash Table Example

- I have 5 friends whose phone numbers I want to store in a hash table of size 8. Their names and number are:

Susy Olson	555-1212
Sarah Skillet	555-4321
Ryan Hamster	545-3241
Justin Case	555-6798
Chris Lindmeyer	535-7869

- I use as a hash function $h(k) = k \bmod 8$, where k is the phone number viewed as a 7-digit decimal number.
- Notice that

$$5554321 \bmod 8 = 5453241 \bmod 8 = 1.$$

  But I can't put *Sarah Skillet* and *Ryan Hamster* both in the position 1.

- Can we fix this problem?

Hash Table Problems

- The problem with hash tables is that two keys can have the same hash value. This is called **collision**.
- There are several ways to deal with this problem:
  - Pick the hash function h to minimize the number of collisions.
  - Implement the hash table in a way that allows keys with the same hash value to all be stored.
- The second method is almost always needed, even for very good hash functions. Why?
- We will talk about 2 collision resolution techniques:
  - Chaining
  - Open Addressing

Hash Functions

- Most hash functions assume the keys come from the set $\{0, 1, \ldots\}$.
- If the keys are not natural numbers, some method must be used to convert them.
  - Phone number and ID numbers can be converted by removing the hyphens.
  - Characters can be converted using ASCII.
  - Strings can be converted by converting each character using ASCII, and then interpreting the string of natural numbers as if it where stored base 128.
- We need to choose a good hash function.
- A hash function is good if
  - it can be computed quickly, and
  - the keys are distributed uniformly throughout the table.

Some Good Hash Functions

- **The division method:**

$$h(k) = k \bmod m$$

  - The modulus m must be chosen carefully.
  - Powers of 2 and 10 can be bad. Why?
  - Prime numbers not too close to powers of 2 are a good choice.
  - We can pick m by choosing an appropriate prime number that is close to the table size we want.

- **The multiplication method:** Let A be a constant with $0 < A < 1$. Then we use

$$h(k) = \lfloor m(kA \bmod 1) \rfloor,$$

where "$kA \bmod 1$" means the fractional part of kA.

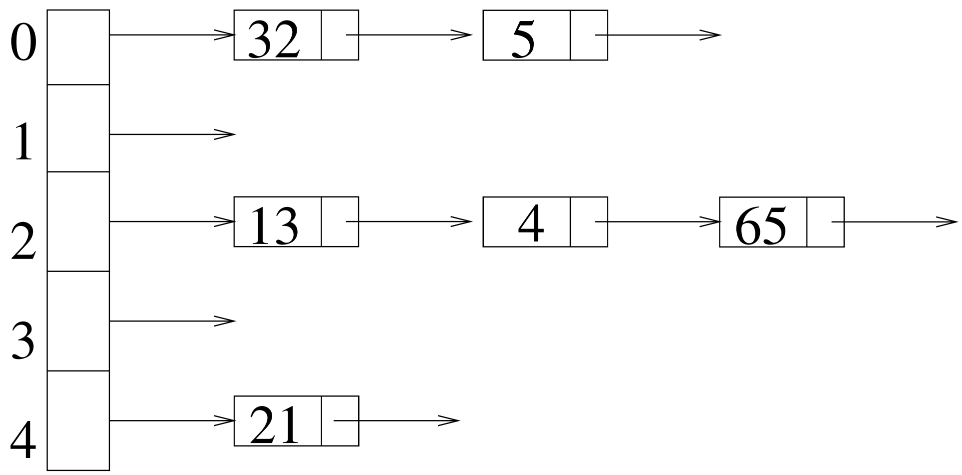
  - The choice of m is not as critical here.
  - We can choose m to make the implementation easy and/or fast.

Universal Hashing

- Let x and y be distinct keys.
- Let $\mathcal{H}$ be a set of hash functions.
- $\mathcal{H}$ is called **universal** if the number of functions $h \in \mathcal{H}$ for which $h(x) = h(y)$ is precisely $|\mathcal{H}|/m$.
- In other words, if we pick a random $h \in \mathcal{H}$, the probability that x and y collide under h is $1/m$.
- Universal hashing can be useful in many situations.
  - You are asked to come up with a hashing technique for your boss.
  - Your boss tells you that *after* you are done, he will pick some keys to hash.
  - If you get too many collisions, he will fire you.
  - If you use universal hashing, the only way he can succeed is by getting lucky. Why?

Collision Resolution: Chaining

- With *chaining*, we set up an array of links, indexed by the hash values, to lists of items with the same hash value.



- Let n be the number of keys, and m the size of the hash table. Define the **load factor** $\alpha = n/m$.
- Successful and unsuccessful searching both take time $\Theta(1+\alpha)$ on average, assuming simple uniform hashing.
- By **simple uniform hashing** we mean that a key has equal probability to hash into any of the m slots, independent of the other elements of the table.

Collision Resolution: Open Addressing

- With Open Addressing, all elements are stored in the hash table.
- This means several things
  - Less memory is used than with chaining since we don't have to store pointers.
  - The hash table has an absolute size limit of m. Thus, planning ahead is important when using open addressing.
  - We must have a way to store multiple elements with the same hash value.
- Instead of a hash function, we need to use a **probe sequence**. That is, a sequence of hash values.
- We go through the sequence one by one until we find an empty position.
- For searching, we do the same thing, skipping values that do not match our key.

Probe Sequences

- We define our hash functions with an extra parameter, the probe number i. Thus our hash functions look like $h(k, i)$.
- We compute $h(k, 0)$, $h(k, 1)$, ..., $h(k, i)$ until $h(k, i)$ is empty.
- The hash function h must be such that the **probe sequence** $\langle h(k, 0), \ldots h(k, m-1)\rangle$ is a permutation of $\langle 0, 1, \ldots, m-1\rangle$. Why?
- Insertion and searching are fairly quick, assuming a good implementation.
- Deletion can be a problem because the probe sequence can be complex.
- We will discuss 2 ways of defining probe sequences:
  - Linear Probing
  - Double Hashing

Linear Probing

- Let g be a hash function.
- When we use linear probing, the probe sequence is computed by

$$h(k,i) = (g(k)+i) \bmod m.$$

- When we insert an element, we start with the hash value, and proceed element by element until we find an empty slot.
- For searching, we start with the hash value and proceed element by element until we find the key we are looking for.
- **Example:** Let $g(k) = k \bmod 13$. We will insert the following keys into the hash table:

$$18, 41, 22, 44, 59, 32, 31, 73$$

0	1	2	3	4	5	6	7	8	9	10	11	12

- Problem: The values in the table tend to *cluster*.

Double Hashing

- With **double hashing** we use two hash functions.
- Let h_1 and h_2 be hash functions.
- We define our probe sequence by

$$h(k, i) = (h_1(k) + i \times h_2(k)) \bmod m.$$

- We must ensure that m and $h_2(k)$ are relatively prime for all values of k. Why?
- One method to do this is to pick $m = 2^n$ for some n, and ensure that $h_2(k)$ is always odd.
- Double hashing tends to distribute keys more uniformly than linear probing.

Double Hashing Example

- Let $h_1 = k \bmod 13$
- Let $h_2 = 1 + (k \bmod 8)$, and
- Let the hash table have size 13.
- Then our probe sequence is defined by

$$h(k, i) = (h_1(k) + i \times h_2(k)) \bmod 13$$

$$= (k \bmod 13 + i(1 + (k \bmod 8))) \bmod 13$$

- Insert the following keys into the table:

$$18, 41, 22, 44, 59, 32, 31, 73$$

0	1	2	3	4	5	6	7	8	9	10	11	12

Open Addressing: Performance

- Again, we set $\alpha = n/m$. This is the average number of keys per array index.
- Note that $\alpha \leq 1$ for open addressing.
- We assume a good probe sequence has been used. That is, for any key, each permutation of $\langle 0, 1 \ldots, m-1 \rangle$ is equally likely as a probe sequence.
- The average number of probes for insertion or unsuccessful search is at most $1/(1-\alpha)$.
- The average number of probes for a successful search is at most

$$\frac{1}{\alpha} \ln \frac{1}{1-\alpha} + \frac{1}{\alpha}.$$

Chaining or Open Addressing?