DNF Suppliment

Definition (DNF)

A propositional formula is in **disjunctive normal form (DNF)** if it is a disjunction (OR) of one or more *terms*, where each term is a conjunction (AND) of *literals*. A **literal** is either a variable (e.g., p) or its negation (e.g., $\neg p$).

Equivalently, a formula is in DNF if it has the form

$$T_1 \ \vee \ T_2 \ \vee \ \cdots \ \vee \ T_k,$$

where each T_i has the form

$$T_i = (\ell_1 \ \wedge \ \ell_2 \ \wedge \ \cdots \ \wedge \ \ell_m),$$

and where each ℓ_j is a literal (that is, a variable or its negation (i.e. p or $\neg p$ for some variable p)).

It is important to note that both k and m are allowed to be 1. Thus:

- A single literal such as p is in DNF.
- A single conjunction such as $p \wedge q$ is in DNF.
- A disjunction such as $p \vee q$ is in DNF.

(Think about it this way in terms of multiplication and addition: If I say a sum is the addition of one or more numbers, then clearly $1 + 2 + 3$ is a sum, but so is 1. Similarly, $4 \cdot 3 \cdot 2$ is the multiplication of three numbers, and 5 can be viewed as the multiplication of a single number, even though there is nothing else to multiply by.)

In short, DNF means "OR of ANDs of literals," where a single conjunctive term (e.g., $p \wedge q$) is allowed, and a single literal (e.g., p) is allowed to count as a conjunction of length 1.

Examples that *are* in DNF

1. p This is a single literal, hence a conjunction of length 1 (a term). So it is a disjunction of one term, and therefore DNF. $\neg p$ is also in DNF for the same reason.

2. $p \wedge q$ This is a conjunction of literals (p and q), so it is a single term. A single term is allowed, so the whole formula is in DNF.

3. $p \vee q$ This is a disjunction of two terms: the term p and the term q. Each of p and q is a literal (thus a 1-literal conjunction), so the formula is in DNF.

4. $\neg p \wedge q$ This is a conjunction of literals: $\neg p$ is a literal and q is a literal. So it is a single term, hence DNF.

5. $\neg p \vee q$ This is a disjunction of two terms: $\neg p$ and q. Each is a literal (thus a term), so the formula is in DNF.

6. $(\neg p \wedge q) \ \vee \ (r \wedge p \wedge \neg q)$ This is a disjunction of two terms.

- The left term $(\neg p \wedge q)$ is a conjunction of literals.
- The right term $(r \wedge p \wedge \neg q)$ is also a conjunction of literals.

Since it is an OR of ANDs of literals, it is in DNF.

Examples that ***are not*** in DNF

Key reason

DNF requires that negation $\neg$ apply only to individual variables (literals), not to a compound formula. It also requires that the top-level structure be an OR of AND-clauses of literals (no OR nested inside an AND without distributing).

1. $\neg(p \land q)$ This is *not* DNF because the negation applies to the compound subformula $(p \land q)$, not to a single variable. In DNF, negation is only allowed directly on variables (literals).

2. $\neg(p \lor q)$ This is *not* DNF for the same reason: the negation applies to the compound subformula $(p \lor q)$, not to a single variable.

3. $(p \lor q) \land (q \land r)$ This is *not* DNF because it has a conjunction where one side is a disjunction: $(p \lor q)$ is an OR, and it is being ANDed with something else. DNF requires an *OR of ANDs*; here we have an *AND* whose left part is an OR, so it is not in DNF as written.

Additional Examples of Formulas that Are *Not* in DNF

1. Negation applied to a compound formula

These are not in DNF because $\neg$ is applied to something other than a single variable.

$$\neg(p \land q), \qquad \neg(p \lor q), \qquad \neg(p \land (q \lor r))$$

2. Disjunction inside a conjunction

These are not in DNF because an OR appears inside an AND, violating the "OR of ANDs" structure.

$$(p \lor q) \land r, \qquad p \land (q \lor r), \qquad (p \lor q) \land (r \lor s)$$

3. Mixed structure with no top-level disjunction

These formulas contain both $\lor$ and $\land$, but the overall shape is not a disjunction of conjunctive terms.

$$(p \land (q \lor r)) \lor s, \qquad (p \lor (q \land r)) \lor s \qquad p \land (q \land r \lor s)$$

Conjunctive Normal Form (CNF)

Disjunctive clauses. A *disjunctive clause* is a disjunction (OR) of one or more literals. Examples of disjunctive clauses include:

$$p \lor q, \qquad \neg p \lor r, \qquad q.$$

Notice that these look a lot like conjuctive clauses. They have the same form if you just replace AND with OR.

Conjunctive normal form (CNF). A logical expression is in *conjunctive normal form (CNF)* if it is expressed as a conjunction (AND) of one or more disjunctive clauses.

Examples of CNF. Each of the following formulas is in CNF:

$$p, \qquad p \lor q, \qquad (p \lor q) \land (\neg p \lor r),$$

$$(p \lor q \lor r) \ \land \ (\neg p \lor s) \ \land \ (q \lor \neg r),$$

$$(\neg p \lor q) \ \land \ (p \lor r \lor s) \ \land \ (\neg q \lor \neg s),$$

$$(p \lor q) \ \land \ (\neg p \lor r) \ \land \ (q \lor s \lor \neg r).$$

Contrast with DNF. For comparison, the following formulas are in disjunctive normal form (DNF), since each is an OR of conjunctive clauses:

$$p, \qquad p \land q, \qquad (p \land q) \lor (\neg p \land r),$$

$$(p \land q \land r) \ \lor \ (\neg p \land s),$$

$$(\neg p \land q) \ \lor \ (p \land r \land s) \ \lor \ (\neg q \land \neg s),$$

$$(p \land q) \ \lor \ (\neg p \land r) \ \lor \ (q \land s \land \neg r).$$

Formulas that are both CNF and DNF. Some formulas satisfy the definitions of *both* conjunctive normal form (CNF) and disjunctive normal form (DNF). This happens when the formula contains only one kind of clause.

Example 1: p. The formula p is a single literal.

- As CNF: p is a disjunction of one literal, and therefore a single disjunctive clause. A conjunction of one disjunctive clause is allowed, so p is in CNF.
- As DNF: p is a conjunction of one literal, and therefore a single conjunctive clause. A disjunction of one conjunctive clause is allowed, so p is in DNF.

Example 2: $p \land q$. The formula $p \land q$ is a conjunction of literals.

- As DNF: $p \land q$ is a single conjunctive clause. A disjunction consisting of one conjunctive clause is allowed, so it is in DNF.
- As CNF: the entire formula is already a conjunction. Each conjunct must be a disjunctive clause, and a single literal counts as a disjunctive clause of length 1. Thus $p \land q$ is also in CNF.

Example 3: $p \vee q$**.** The formula $p \vee q$ is a disjunction of literals.

- As CNF: $p \vee q$ is a single disjunctive clause. A conjunction consisting of one disjunctive clause is allowed, so it is in CNF.
- As DNF: the entire formula is already a disjunction. Each disjunct must be a conjunctive clause, and a single literal counts as a conjunctive clause of length 1. Thus $p \vee q$ is also in DNF.

Exercises: Disjunctive Normal Form

There are solutions to these exercises on the next page so you can try them and check your work.

Exercise 1: Explaining DNF

For each of the following logical expressions, determine whether it is in disjunctive normal form (DNF). In each case, *explain your reasoning* by referring to the definition of DNF.

1. $(p \wedge q) \ \vee \ (\neg p \wedge r)$
2. $(\neg p \wedge q \wedge r) \ \vee \ (p \wedge \neg r)$
3. $(p \vee q) \wedge r$
4. $(p \wedge (q \vee r)) \ \vee \ s$
5. $\neg(p \wedge q) \ \vee \ r$

Exercise 2: Identifying DNF

Each of the following expressions is either in disjunctive normal form (DNF) or not. Exactly **four** of them are in DNF. Determine which are in DNF and which are not.

1. $(p \wedge q \wedge r) \ \vee \ (\neg p \wedge s)$
2. $(p \vee q) \ \vee \ (\neg r)$
3. $(p \vee q) \ \wedge \ (\neg p \vee r)$
4. $(\neg p \wedge q) \ \vee \ r$
5. $(p \vee q) \ \wedge \ r$
6. $p \ \vee \ (q \wedge r)$
7. $(p \wedge \neg q) \ \vee \ (r \wedge s \wedge \neg p)$
8. $\neg p \ \vee \ \neg(q \wedge r)$

Solutions

Exercise 1: Explaining DNF

Recall: a formula is in DNF if it is an OR of one or more *conjunctive clauses*, and each conjunctive clause is an AND of one or more literals (variables or their negations).

(1) $(p \wedge q) \ \vee \ (\neg p \wedge r)$
In DNF. It is a disjunction of two terms. Each term is a conjunction of literals: $p \wedge q$ and $\neg p \wedge r$.

(2) $(\neg p \wedge q \wedge r) \ \vee \ (p \wedge \neg r)$
In DNF. It is a disjunction of two terms, and each term is a conjunction of literals.

(3) $(p \vee q) \wedge r$
Not in DNF. The formula is an AND whose left side is an OR. DNF requires an OR of AND-clauses; here an OR occurs inside a conjunction.

(4) $(p \wedge (q \vee r)) \ \vee \ s$
Not in DNF. Although the top-level connective is OR, the left disjunct $p \wedge (q \vee r)$ is not a conjunction of literals because it contains the compound subformula $(q \vee r)$ inside an AND.

(5) $\neg(p \wedge q) \ \vee \ r$
Not in DNF. In DNF, negation may only apply directly to a variable. Here $\neg$ is applied to the compound formula $(p \wedge q)$.

Exercise 2: Identifying DNF

(1) $(p \wedge q \wedge r) \ \vee \ (\neg p \wedge s)$
DNF. OR of two conjunctions of literals.

(2) $(p \vee q) \ \vee \ (\neg r)$
Not DNF. The term $(p \vee q)$ is not a conjunction of literals; it is a disjunction. (A DNF disjunct must be a conjunction of literals, e.g. $p \wedge q \wedge \neg r$.)

(3) $(p \vee q) \ \wedge \ (\neg p \vee r)$
Not DNF. This is an AND of disjunctions (it is in CNF, not DNF).

(4) $(\neg p \wedge q) \ \vee \ r$
DNF. OR of two terms: $(\neg p \wedge q)$ is a conjunction of literals, and r is a single literal (a conjunction of length 1).

(5) $(p \vee q) \ \wedge \ r$
Not DNF. The formula is a conjunction whose left conjunct $(p \vee q)$ is a disjunction rather than a conjunction of literals. Since DNF requires an OR of conjunctive clauses, having an OR inside an AND violates the required structure.

(6) $p \ \vee \ (q \wedge r)$
DNF. OR of two terms: p is a literal (a conjunction of length 1) and $(q \wedge r)$ is a conjunction of literals.

(7) $(p \wedge \neg q) \ \vee \ (r \wedge s \wedge \neg p)$
DNF. OR of two conjunctions of literals.

(8) $\neg p \;\vee\; \neg(q \wedge r)$
Not DNF. The second disjunct is $\neg(q \wedge r)$, where negation is applied to a compound formula rather than directly to a variable.

Thus, the four DNF expressions in Exercise 2 are **1, 4, 6, 7**.