Equivalence Relations

Definition: A relation $R$ on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive. Recall the definitions:

- **reflexive:** $(a, a) \in R$ for all $a \in A$.
- **symmetric:** $(a, b) \in R$ when $(b, a) \in R$, for $a, b \in A$.
- **transitive:** $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, for $a, b, c \in A$.

If two elements are related by an equivalence relation, they are said to be equivalent.
Examples

1. Let $R$ be the relation on the set of English words such that $\alpha R \beta$ if and only if $\alpha$ starts with the same letter as $\beta$. Then $R$ is an equivalence relation.

2. Let $R$ be the relation on the set of all human beings such that $x R y$ if and only if $x$ was born in the same country as $y$. Then $R$ is an equivalence relation.

3. Let $R$ be the relation on the set of all human beings such that $x R y$ if and only if $x$ owns the same color car as $y$. Then $R$ is an *not* equivalence relation.
Let $m > 1$ be a positive integer. Then the relation

$$R = \{(a, b) : a \equiv b \mod m\}$$

is an equivalence relation.

**Proof:** By definition, $a \equiv b \mod m$ if and only if $a - b = mk$, for some integer $k$. Using this, we proceed:

- Since $a - a = 0 = 0m$, we have that $a \equiv a \mod m$, and $R$ is reflexive.

- If $a \equiv b \mod m$, then $a - b = km$, for some integer $k$. Thus, $b - a = (-k)m$, and we have $b \equiv a \mod m$, so $R$ is symmetric.
• If \( a \equiv b \mod m \), and \( b \equiv c \mod m \), then we have 
\[
a - b = km \quad \text{and} \quad b - c = lm,
\]
for integers \( k \) and \( l \). Thus,
\[
a - c = (a - b) + (b - c) = km + lm = (k + l)m,
\]
and we have \( a \equiv c \mod m \), and \( R \) is transitive.

Therefore, congruence modulo \( m \) is an equivalence relation \( \blacksquare \)
Definition: Let $R$ be an equivalence relation on a set $A$. The *equivalence class* of $a$ is

$$[a]_R = \{ b : (a, b) \in R \}.$$ 

In words, $[a]_R$ is the set of all elements that are related to the element $a \in A$. If the relation is clear, we can omit the subscript (i.e. $[a]$ instead of $[a]_R$). If $b \in [a]_R$, then $b$ is called a *representative* of the equivalence class.
Examples Continued

1. The equivalence class of *Xenon* is all words starting with the letter $X$. That is,

$$[\text{Xenon}] = \{ \alpha : \alpha \text{ is an English word starting with the letter } X \}$$

2. The equivalence class of *Chuck Cusack* is all people born in the United States of America. That is,

$$[\text{Chuck Cusack}] = \{ A : A \text{ is a person that was born in the U.S.A.} \}$$
Example: Congruence Classes Modulo $m$

The congruence class of an integer $a$ modulo $m$ is denoted by $[a]_m$.

Thus,

$$[3]_5 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$

$$[0]_8 = \{\ldots, -16, -8, 0, 8, 16, \ldots\}$$

$$[5]_4 = [1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$
Equivalence Classes and Partitions

Theorem 1: Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

1. $aRb$
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$

Proof: Show that $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$.

Notice that this theorem says that if the intersection of two equivalence classes is not empty, then they are equal. That is, two equivalence classes are either equal or disjoint.
Definition: A partition of a set $S$ is a collection of disjoint nonempty subsets of $S$ whose union is $S$. That is, a partition of $S$ is a collection of subsets $A_i, i \in I$ such that

$$A_i \neq \emptyset \text{ for } i \in I,$$

$$A_i \cap A_j = \emptyset, \text{ when } i \neq j, \text{ and}$$

$$\cup_{i \in I} A_i = S.$$  

($I$ is an index set. For example, often $I = \{1, 2, \ldots, n\}$.)
**Theorem 2:** Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i : i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i, i \in I$, as its equivalence classes.

**Proof (informal):** The equivalence classes of an equivalence relation are nonempty (since $a \in [a]_R$), and by Theorem 1 are disjoint. Since every element of the set $S$ is in some equivalence class (e.g. $a \in [a]_R$), the equivalence classes partition $S$. 
(Proof of Theorem 2, continued)
Now, assume we have a partition \( \{ A_i : i \in I \} \) of a set \( S \).
Define a relation on \( S \) by \( aRb \) if and only if \( a, b \in A_i \) for some \( i \). It is not hard to see that this is an equivalence relation.

**Example:** We can partition the set of integers according to the equivalence classes modulo 5 as follows:

\[
[0]_5 = \{ \ldots, -10, -5, 0, 5, 10, \ldots \},
\]
\[
[1]_5 = \{ \ldots, -9, -4, 1, 6, 11, \ldots \},
\]
\[
[2]_5 = \{ \ldots, -8, -3, 2, 7, 12, \ldots \},
\]
\[
[3]_5 = \{ \ldots, -7, -2, 3, 8, 13, \ldots \},
\]
\[
[4]_5 = \{ \ldots, -6, -1, 4, 9, 14, \ldots \}.
\]
Example: Let $R$ be the equivalence relation on the set of English words defined by $\alpha R \beta$ if and only if $\alpha$ starts with the same letter as $\beta$. Then we can partition the set of English words as follows:

\[
\begin{align*}
[a] &= \{a, ant, any, able, \ldots\}, \\
[bread] &= \{be, big, bread, brown, \ldots\}, \\
\ldots \\
\ldots \\
\ldots \\
[zip] &= \{zebra, zed, zip, zoo, \ldots\}.
\end{align*}
\]