# Equivalence Relations

**Definition:** A relation $R$ on a set $A$ is called an *equivalence relation* if it is reflexive, symmetric, and transitive. Recall the definitions:

- **reflexive:** $(a, a) \in R$ for all $a \in A$.
- **symmetric:** $(a, b) \in R$ when $(b, a) \in R$, for $a, b \in A$.
- **transitive:** $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$, for $a, b, c \in A$.

If two elements are related by an equivalence relation, they are said to be *equivalent*.

## Examples

1. Let $R$ be the relation on the set of English words such that $\alpha R\beta$ if and only if $\alpha$ starts with the same letter as $\beta$. Then $R$ is an equivalence relation.

2. Let $R$ be the relation on the set of all human beings such that $xRy$ if and only if $x$ was born in the same country as $y$. Then $R$ is an equivalence relation.

3. Let $R$ be the relation on the set of all human beings such that $xRy$ if and only if $x$ owns the same color car as $y$. Then $R$ is an *not* equivalence relation.

# Congruence Modulo $m$

Let $m > 1$ be a positive integer. Then the relation

$$R = \{(a, b) : a \equiv b \bmod m\}$$

is an equivalence relation.

**Proof:** By definition, $a \equiv b \bmod m$ if and only if $a - b = mk$, for some integer $k$. Using this,we proceed:

- Since $a - a = 0 = 0m$, we have that $a \equiv a \bmod m$, and $R$ is reflexive.
- If $a \equiv b \bmod m$, then $a - b = km$, for some integer $k$. Thus, $b - a = (-k)m$, and we have $b \equiv a \bmod m$, so $R$ is symmetric.

- If $a \equiv b \bmod m$, and $b \equiv c \bmod m$, then we have $a - b = km$ and $b - c = lm$, for integers $k$ and $l$. Thus,

  $$a - c = (a - b) + (b - c) = km + lm = (k + l)m,$$

  and we have $a \equiv c \bmod m$, and $R$ is transitive.

Therefore, congruence modulo $m$ is an equivalence relation ■

**Definition:** Let $R$ be an equivalence relation on a set $A$. The *equivalence class of* $a$ is

$$[a]_R = \{b : (a, b) \in R\}.$$

In words, $[a]_R$ is the set of all elements that are related to the element $a \in A$. If the relation is clear, we can omit the subscript (i.e. $[a]$ instead of $[a]_R$).
If $b \in [a]_R$, then $b$ is called a *representative* of the equivalence class.

# Examples Continued

1. The equivalence class of **Xenon** is all words starting with the letter *X*. That is,

$$[\text{Xenon}] = \{\alpha : \alpha \text{ is an English word starting with the letter } X\}$$

2. The equivalence class of *Chuck Cusack* is all people born in the United States of America. That is,

$$[\text{Chuck Cusack}] = \{A : A \text{ is a person that was } \textit{born in the U.S.A.}\}$$

## Example: Congruence Classes Modulo $m$

The congruence class of an integer $a$ modulo $m$ is denoted by $[a]_m$.

Thus,

$$[3]_5 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$

$$[0]_8 = \{\ldots, -16, -8, 0, 8, 16, \ldots\}$$

$$[5]_4 = [1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

## Equivalence Classes and Partitions

**Theorem 1:** Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

$1. aRb$ $\quad$ $2. [a] = [b]$ $\quad$ $3. [a] \cap [b] \neq \emptyset$

**Proof:** Show that $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$.

Notice that this theorem says that if the intersection of two equivalence classes is not empty, then they are equal. That is, two equivalence classes are either equal or disjoint.

**Definition:** A *partition* of a set $S$ is a collection of disjoint nonempty subsets of $S$ whose union is $S$. That is, a partition of $S$ is a collections of subsets $A_i$, $i \in I$ such that

$A_i \neq \emptyset$ for $i \in I$,

$A_i \cap A_j = \emptyset,$ when $i \neq j,$ and

$\cup_{i \in I} A_i = S.$

($I$ is an index set. For example, often $I = \{1, 2, \ldots, n\}$.)

**Theorem 2:** Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i : i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i$, $i \in I$, as its equivalence classes.

**Proof (informal):** The equivalence classes of an equivalence relation are nonempty (since $a \in [a]_R$), and by Theorem 1 are disjoint. Since every element of the set $S$ is in some equivalence class (e.g. $a \in [a]_R$), the equivalence classes partition $S$.

(Proof of Theorem 2, continued)
Now, assume we have a partition $\{A_i : i \in I\}$ of a set $S$. Define a relation on $S$ by $aRb$ if and only if $a, b \in A_i$ for some $i$. It is not hard to see that this is an equivalence relation

■

**Example:** We can partition the set of integers according to the equivalence classes modulo 5 as follows:

$$[0]_5 = \{\ldots, -10, -5, 0, 5, 10, \ldots\},$$

$$[1]_5 = \{\ldots, -9, -4, 1, 6, 11, \ldots\},$$

$$[2]_5 = \{\ldots, -8, -3, 2, 7, 12, \ldots\},$$

$$[3]_5 = \{\ldots, -7, -2, 3, 8, 13, \ldots\},$$

$$[4]_5 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}.$$

**Example:** Let $R$ be the equivalence relation on the set of English words defined by $\alpha R \beta$ if and only if $\alpha$ starts with the same letter as $\beta$. Then we can partition the set of English words as follows:

$$\begin{array}{lcl}
[a] & = & \{a, ant, any, able, \ldots\}, \\
[bread] & = & \{be, big, bread, brown, \ldots\}, \\
 & . & \\
 & . & \\
 & . & \\
[zip] & = & \{zebra, zed, zip, zoo, \ldots\}.
\end{array}$$