Induction Proofs

Chuck Cusack

The *principle of mathematical induction* (*PMI*, or simply *induction*) is usually used to prove statements of the form "for all $n \geq a$, $P(n)$ is true," where a is a constant, usually 0 or 1. To prove a statement using the *PMI*, one needs to prove a *base case*, and then show that if $P(k)$ is true for any k, then $P(k+1)$ is true.

The principle of mathematical induction is based on the rule of inference

$$[P(a) \land \forall n(P(n) \to P(n+1)] \to \forall n P(n),$$

where the universe of discourse is the set of integers $\{a, a+1, a+2, \ldots\}$.

Since induction proofs can be difficult for some people to understand, I have developed a formal way of writing induction proofs that explicitly states every step of the proof. I want to be clear that the formalism given here is not required for a proper proof, but is a tool for understanding and constructing induction proofs. The first several examples will be written using the formalism. The last example gives several alternative ways you might see an induction proof written.

The steps I suggest for learning how to construct an induction proof are as follows:

1. **Define:** Define $P(n)$ based on the statement.
   $P(n)$ should be a statement about a single instance, not about a series of instances. For example, it should be statements like "$2n$ is even" or "A set with n elements has 2^n subsets," *NOT* of the form "$2n$ is even if $n > 1$," "$n^2 > 0$ if $n \neq 0$," or "For all $n > 1$, a set with n elements has 2^n subsets."

2. **Rephrase:** Rephrase the statement using $P(n)$.
   In almost all cases, the rephrased statement should be "For all $n \geq a$, $P(n)$ is true," where a is some constant, often 0 or 1. If the statement cannot be phrased in this way, induction may not be appropriate.

3. **Base Case:** Prove the base case or cases.
   For most statements, this means showing that $P(a)$ is true, where a is the value from the rephrased statement. Sometimes one must prove multiple base cases, usually $P(a)$, $P(a+1)$, …, $P(a+i)$ for some $i > 0$. For most statements, 1 or 2 base cases suffice.

4. **Show:** For clarity, explicitly write down what your next step is.
   It is almost always either:
   $$P(k) \to P(k+1) \text{ if } k \geq a,$$
   or
   $$[P(a) \land P(a+1) \land \cdots \land P(k)] \to P(k+1) \text{ if } k \geq a.$$
   The second case is sometimes called the *second principle of mathematical induction (*PMI2*)*. In either case, you are stating that you need to prove that $P(k+1)$ is true, assuming that either $P(k)$ is true, or that $P(a)$, $P(a+1)$, …, $P(k)$ are all true. It is important to note that this step can also be
   $$P(k-1) \to P(k) \text{ if } k > a$$
   if this makes the induction step easier.

5. **Hypothesis:** This is almost always either
   $$P(k) \text{ is true,}$$
   $$P(k-1) \text{ is true,}$$
   or
   $$[P(a) \land P(a+1) \land \cdots \land P(k)] \text{ is true.}$$

6. **Induction:** Given the **Hypothesis**, prove the **Show** statement.
   This is the longest, and most varied, part of the proof.

7. **Summary:** Almost always either:
   "Since we proved that $P(a)$ is true, and that $P(k) \to P(k+1)$, by ***PMI***, $P(n)$ is true for all $n \geq a$."
   or
   "Since we proved that $P(a)$ is true, and that $[P(a) \wedge P(a+1) \wedge \cdots \wedge P(k)] \to P(k+1)$, by ***PMI2***, $P(n)$ is true for all $n \geq a$."

The **Induction** step is the hardest step of the proof. In fact, the other steps are all trivial most of the time, except the **Base Case**, which can be tricky occasionally.

Many statements $P(k)$ are of the form "$LHS(k) = RHS(k)$" (where $=$ might be replaced with $\geq$, $\leq$, etc.). For instance, if $P(k)$ is the statement "$k > 2^k$", $LHS(k) = k$, and $RHS(k) = 2^k$. In these cases, the goal of the induction step is to show that $LHS(k+1) = RHS(k+1)$ given that $LHS(k)$=$RHS(k)$. The way this is usually done is as follows:

$$\begin{aligned} LHS(k+1) &= LHS(k) + \text{stuff} && \text{applying algebra} \\ &= RHS(k) + \text{stuff} && \text{by hypothesis (since } P(k) \text{ is true)} \\ &= \cdots && \text{1 or most steps, usually involving algebra} \\ &= RHS(k+1) && \text{more algebra, resulting in the goal} \end{aligned}$$

You will see that several of the examples in these notes will follow this pattern, including the first example, which we now present.

Example 1:
Prove that the sum of the first n odd integers is n^2. That is, show that $\sum_{i=1}^{n}(2i-1) = n^2$ for all $n \geq 1$.
Proof:

1. **Define:** Let $P(n)$ be the statement "$\sum_{i=1}^{n}(2i-1) = n^2$".

2. **Rephrase:** $P(n)$ is true for all $n \geq 1$.

3. **Base Case:** Since $\sum_{i=1}^{1}(2i-1) = 2 \cdot 1 - 1 = 1 = 1^2$, $P(1)$ is true.

4. **Show:** $P(k) \to P(k+1)$ if $k \geq 1$.

5. **Hypothesis:** $P(k)$ is true.

6. **Induction:** Notice that

$$\begin{aligned} \sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) && \text{write as } LHS(k)\text{+stuff} \\ &= k^2 + (2k+2-1) && \text{since } P(k) \text{ is true} \\ &= k^2 + 2k + 1 && \text{algebra} \\ &= (k+1)^2 && \text{algebra} \end{aligned}$$

   Thus $P(k+1)$ is true.

7. **Summary:** Since we proved that $P(1)$ is true, and that $P(k) \to P(k+1)$, by ***PMI***, $P(n)$ is true for all $n \geq 1$.

Example 2:
Prove that $n < 2^n$ for all integers $n \geq 1$.
Proof:

1. **Define:** Let $P(n)$ be the statement "$n < 2^n$".

2. **Rephrase:** We want to prove that $P(n)$ is true for all $n \geq 1$.

3. **Base Case:** Since $1 < 2^1$, $P(1)$ is clearly true.

4. **Show:** We need to show that $P(k) \rightarrow P(k+1)$ if $k \geq 1$.

5. **Hypothesis:** We assume $P(k)$ is true.

6. **Induction:** By hypothesis (since $P(k)$ is true), we know that $k < 2^k$, thus

$$\begin{array}{rcll} k+1 & < & 2^k + 1 & \text{adding 1 to both sides of previous} \\ & < & 2^k + 2^k & \text{since } 1 < 2^k \text{ when } k \geq 1 \\ & = & 2(2^k) & \text{algebra} \\ & = & 2^{k+1} & \text{algebra} \end{array}$$

   Since we have shown that $k+1 < 2^{k+1}$, $P(k+1)$ is true.

7. **Summary:** Since we proved that $P(1)$ is true, and that $P(k) \rightarrow P(k+1)$, by ***PMI***, $P(n)$ is true for all $n \geq 1$.

The next example requires the second principle of mathematical induction.

Example 3:
Show that every integer $n \geq 2$ can be written as the product of primes.
Proof:

1. **Define:** Let $P(n)$ be the statement "n can be written as the product of primes."

2. **Rephrase:** For all $n \geq 2$, $P(n)$ is true.

3. **Base Case:** Since 2 is clearly prime, it can be written as the product of one prime. Thus $P(2)$ is true.

4. **Show:** We need to show that $[P(2) \wedge P(3) \wedge \cdots \wedge P(k-1)] \rightarrow P(k)$ if $k \geq 2$.

5. **Hypothesis:** We assume $[P(2) \wedge P(3) \wedge \cdots \wedge P(k-1)]$ is true.

6. **Induction:** If k is prime, clearly $P(k)$ is true. If k is not prime, then we can write $k = a \cdot b$, where $2 \leq a \leq b < k$. By hypothesis (since $P(a)$ and $P(b)$ are true), a and b can be written as the product of primes. Therefore, k can be written as the product of primes, namely the primes from the factorizations of a and b. Thus $P(k)$ is true.

7. **Summary:** Since we proved that $P(2)$ is true, and that $[P(2) \wedge P(3) \wedge \cdots \wedge P(k-1)] \rightarrow P(k)$ if $k \geq 2$, by the ***PMI2***, $P(n)$ is true for all $n \geq 2$.

We present 7 proofs of the next example. Proofs 2 through 4 are appropriate in most instances. Proofs 6 and 7 are the type you might find in a textbook or journal article.

Example 4:
Prove the generalized form of DeMorgan's law. That is, show that for any $n \geq 2$, if p_1, p_2, ..., p_n are propositions, then $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $(\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$.
Proof #1:

1. **Define:** Let $P(n)$ be the statement "$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$."

2. **Rephrase:** For all $n \geq 2$, $P(n)$ is true.

3. **Base Case:** $P(2)$ is DeMorgan's law, which is clearly true.

4. **Show:** We need to show that $P(k) \rightarrow P(k+1)$ if $k \geq 2$.

5. **Hypothesis:** We assume $P(k)$ is true.

6. **Induction:** Notice that

$$\begin{array}{rcll} \neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) & \Leftrightarrow & \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) & \text{associative law} \\ & \Leftrightarrow & \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1} & \text{DeMorgan's law} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} & \text{hypothesis} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1}) & \text{associative law} \end{array}$$

Thus $P(k+1)$ is true.

7. **Summary:** Since we proved that $P(2)$ is true, and that $P(k) \to P(k+1)$ if $k \geq 2$, by the ***PMI***, $P(n)$ is true for all $n \geq 2$.

Proof #2:
Let $P(n)$ be the statement "$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$." The we want to show that for all $n \geq 2$, $P(n)$ is true. $P(2)$ is DeMorgan's law, so the base case is true. Assume $P(k)$ is true. Then notice that

$$\begin{array}{rcll} \neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) & \Leftrightarrow & \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) & \text{associative law} \\ & \Leftrightarrow & \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1} & \text{DeMorgan's law} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} & \text{hypothesis} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1}) & \text{associative law} \end{array}$$

Thus $P(k+1)$ is true. Since we proved that $P(2)$ is true, and that $P(k) \to P(k+1)$ if $k \geq 2$, by the ***PMI***, $P(n)$ is true for all $n \geq 2$.

Proof #3:
We know that $\neg(p_1 \vee p_2) \Leftrightarrow (\neg p_1 \wedge \neg p_2)$ since this is simply DeMorgan's law. Assume the statement is true for k. That is, $\neg(p_1 \vee p_2 \vee \cdots \vee p_k) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k)$. Then we can see that

$$\begin{array}{rcll} \neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) & \Leftrightarrow & \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) & \text{associative law} \\ & \Leftrightarrow & \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1} & \text{DeMorgan's law} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} & \text{hypothesis} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1}) & \text{associative law} \end{array}$$

Thus the statement is true for $k+1$. Since we have shown that the statement is true for $n = 2$, and that whenever it is true for k it is true for $k+1$, by the ***PMI***, the statement is true for all $n \geq 2$.

Proof #4: The case $k = 2$ is DeMorgan's law. Assume the statement is true for k. Then

$$\begin{array}{rcll} \neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) & \Leftrightarrow & \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) & \text{associative law} \\ & \Leftrightarrow & \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1} & \text{DeMorgan's law} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} & \text{hypothesis} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1}) & \text{associative law} \end{array}$$

Thus the statement is true for $k+1$. Since we have shown that the statement is true for $n = 2$, and that whenever it is true for k it is true for $k+1$, by the ***PMI***, the statement is true for all $n \geq 2$.

Proof #5: The case $k = 2$ is DeMorgan's law. Assume the statement is true for k. Then

$$\begin{array}{rcll} \neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) & \Leftrightarrow & \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) & \text{associative law} \\ & \Leftrightarrow & \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1} & \text{DeMorgan's law} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} & \text{hypothesis} \\ & \Leftrightarrow & (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1}) & \text{associative law} \end{array}$$

Thus the statement is true for $k+1$. By the ***PMI***, the statement is true for all $n \geq 2$.

Proof #6: The case $k = 2$ is DeMorgan's law, and if the statement is true for k, then

$$\neg(p_1 \vee \cdots \vee p_{k+1}) \Leftrightarrow \neg(p_1 \vee \cdots \vee p_k) \wedge \neg p_{k+1} \Leftrightarrow (\neg p_1 \wedge \cdots \wedge \neg p_k) \wedge \neg p_{k+1} \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1})$$

Thus the statement is true for $k+1$. By the ***PMI***, the statement is true for all $n \geq 2$.

Proof #7: The result easily follows by induction.