Solving Recurrence Relations

Some Examples

1. Give an asymptotic upper bound for the recurrence $T(n) = 7T(n/2) + 15n^2/4$.
   Solution: We try the *Master Theorem*. We compute

   $$\log_b a = \log_2 7 = \log_2(4(7/4)) = \log 4 + \log(7/4) = 2 + \log_2(7/4).$$

   Then $2 = \log_2 7 - \log_2(7/4)$. If we pick $\epsilon = \log_2(7/4) > 0$, then clearly

   $$15n^2/4 = O(n^2) = O(n^{\log_2 7 - \log_2(7/4)}) = O(n^{\log_b a - \epsilon}),$$

   so case 1 applies, and $T(n) = O(n^{\log_2 7}) \cong O(n^{2.8})$.

2. Give a tight bound for the recurrence $T(n) = T(\sqrt{n}) + 1$
   Solution: We can see that

   $$\begin{aligned} T(n) &= T(n^{1/2}) + 1 \\ &= T(n^{1/4}) + 1 + 1 \\ &= T(n^{1/8}) + 1 + 1 + 1 \\ &\vdots \\ &= T(2) + 1 + \cdots + 1 + 1. \end{aligned}$$

   We need to count the number of '1's in the sum. Notice that when we have $T\left(n^{1/2^i}\right)$ in the sum, there are i ones. Thus, the last line has i ones, where $n^{1/2^i} = 2$. Taking logs on both sides, we get $(1/2^i)\log n = 1$, or $2^i = \log n$. Again taking logs, we get $i = \log\log n$. We assume that $T(2) = c$ for some constant c, so that $T(n) = c + \log\log n = \Theta(\log\log n)$.

3. Find a good bound for the recurrence $T(n) = 8T(n/4) + n^2 \log n$.
   Solution: We will use the *Master Theorem*. We compute $log_b a = log_4 8 = 1.5$. It is clear that

   $$n^2 \log n = \Omega(n^{1.75}) = \Omega(n^{1.5+.25}),$$

   so case 3 applies. We need to show that there is a $c < 1$ such that

   $$8(n/4)^2 \log(n/4) \le cn^2 \log n.$$

   We can simplify this

   $$(1/2)n^2(\log n - \log 4) \le cn^2 \log n$$

   $$(\log n - 2) \le 2c \log n$$

   $$(1 - 2c)\log n \le 2$$

   If we pick $c = .75$, we have

   $$-(1/2)\log n \le 2,$$

   which is true if $n > 1$. Thus, by case 3, $T(n) = \Theta(n^2 \log n)$.

4. Give a tight bound on the recurrence $T(n) = 4T(n/2) + n^2/\log n$.
Solution:
We try the *Master Theorem*. We see that $\log_b a = \log_4 2 = 2$.

Is $n^2/\log n = O(n^{2-\epsilon})$ for some $\epsilon > 0$? Only if $1/\log n = O(n^{-\epsilon})$, which is true only if $\log n = \Omega(n^{\epsilon})$, which isn't true for any $\epsilon > 0$.

Is $n^2/\log n = \Theta(n^2)$? Only if $1/\log n = \Theta(1)$, which it certainly isn't.

Is $n^2/\log n = \Omega(n^{2+\epsilon})$ for some $\epsilon > 0$? Only if $1/\log n = \Omega(n^{\epsilon})$, which isn't true for any $\epsilon > 0$.

Thus, the *Master Theorem* does not work here. After looking at it for a few minutes, though, it's pretty clear that $T(n) = O(n^2 \log\log n)$ (O.K. maybe it's not clear). We'll prove this by induction. We assume that $T(n) \le cn^2 \log\log n$ for some constance $c > 0$. We assume this holds for $n/2$, and we get

$$\begin{aligned} T(n) &= 4T(n/2) + n^2/\log n \\ &\le 4c(n/2)^2 \log\log(n/2) + n^2/\log n \\ &= cn^2 \log\log(n/2) + n^2/\log n \\ &= n^2(c\log\log(n/2) + 1/\log n) \end{aligned}$$

It can be shown that $c\log\log(n/2) + 1/\log n \le c\log\log n$ if $c \ge 2$. (The details are omitted because they are messy. Just assume this is true.) Then we have

$$\begin{aligned} T(n) &\le n^2(c\log\log(n/2) + 1/\log n) \\ &\le n^2 c\log\log n \\ &\le cn^2 \log\log n \end{aligned}$$

Now we only need to worry about the boundary condition. Notice that $\log\log n \le 0$ if $n \le 2$, and positive otherwise. Thus, we need to pick boundary conditions so that the recurrence does not depend on $T(1)$ and $T(2)$ (You should understand why this is the case). If we use $T(3)$, $T(4)$, and $T(5)$ as our boundary conditions, then all other cases depend on these, and not on $T(1)$ or $T(2)$.

From the recurrence, we see that $T(3) = 4 + 9/\log 3 < 10$, $T(4) = 40$, and $T(5) = 32 + 25/\log 5 < 45$. It is not hard to see that if we pick $c \ge 3$, the bound holds for the boundary conditions.

Now, we have shown that if $c \ge 3$, $T(n) \le cn^2 \log\log n$ for the boundary conditions (n = 3, 4, and 5), and we have proved that if it holds for $n/2$ it holds for n, as long as $c \ge 2$. Therefore, for all $n \ge 3$,

$$T(n) \le 3n^2 \log\log n.$$

Thus $T(n) = O(n^2 \log\log n)$.