Some Sample Proofs

Chuck Cusack

1. *Prove that the sum of two odd integers is even.*

   Proof:
   Let x and y be odd integers. Then we can write $x = 2a+1$ and $y = 2b+1$, for some integers a and b. Then clearly $x + y = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1)$, which is even since $a + b + 1$ is an integer. Thus the sum of two odd integers is even.

2. *Prove that if $5n + 2$ is even, then n is even.*

   Proof by contraposition:
   We will prove the equivalent statement that if n is odd, then $5n + 2$ is odd. Assume n is odd. Then $n = 2k + 1$ for some integer k. Then we have that

   $$\begin{aligned} 5n+2 &= 5(2k+1)+2 \\ &= 10k+5+2 \\ &= 10k+7 \\ &= 2(5k+3)+1 \end{aligned}$$

   Since $5k + 3$ is an integer, this shows that $5n + 2$ is odd.

   Proof by contradiction:
   Assume that $5n + 2$ is even but that n is odd. Since n is odd, $n = 2k + 1$ for some integer k. Therefore

   $$\begin{aligned} 5n+2 &= 5(2k+1)+2 \\ &= 10k+5+2 \\ &= 10k+7 \\ &= 2(5k+3)+1 \end{aligned}$$

   which is odd since $5k + 3$ is an integer. But we assumed that $5n + 2$ was even, which is a contradiction. Therefore our assumption that n is odd must be incorrect, so n is even.

3. *Prove that the square of an odd number is odd.*

   Proof:
   Let x be an odd number. Then we can write $x = 2k + 1$ for some integer k. Then $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, where $m = 2k^2 + 2k$ is an integer. Since we can write $x^2 = 2m + 1$ for some integer m, x^2 is odd.

4. *Prove that if x is irrational, then $1/x$ is irrational.*

   Proof:
   The easiest way to prove this is to prove the contrapositive statement: If $1/x$ is rational, then x is rational. Assume $1/x$ is rational. Then $\frac{1}{x} = \frac{a}{b}$ for some integers a and $b \neq 0$. We know that $1/x \neq 0$ (since otherwise $x \cdot 0 = 1$, which is impossible), so $a \neq 0$. Multiplying both sides of the previous equation by x we get $x\frac{a}{b} = 1$. Now if we multiply both sides by $\frac{b}{a}$ (which we can do since $a \neq 0$), we get $x = \frac{b}{a}$. Since a and b are integers with $a \neq 0$, x is rational.

5. *Prove or disprove that every positive integer can be written as the sum of the squares of two integers.*

   Proof:
   The square of an integer is positive and the smallest 4 squares of integers are 0, 1, 4, and 9. Since $9 > 7$, only 0, 1, and 4 can be used to try to sum to 7. Since no two of these add up to 7, we have a counterexample, so the statement is false.

6. *Prove that if A and B are sets, then $A - B = A \cap \overline{B}$.*

   Proof:
   Let $x \in A - B$. By the definition of set difference ($-$), $x \in A$, and $x \notin B$. If $x \notin B$, then $x \in \overline{B}$ by definition of complement. Since $x \in A$, and $x \in \overline{B}$, $x \in A \cap \overline{B}$ by definition of intersection ($\cap$). Since we have shown that membership in $A - B$ implies membership in $A \cap \overline{B}$, we have shown that $A - B \subseteq A \cap \overline{B}$.
   Now let $x \in A \cap \overline{B}$. Then $x \in A$ and $x \in \overline{B}$ by definition of intersection ($\cap$). Since $x \in \overline{B}$, $x \notin B$ by definition of complement. Since $x \in A$ and $x \notin B$, $x \in A - B$ by definition of difference ($-$). Since we have shown that membership in $A \cap \overline{B}$ implies membership in $A - B$, we have shown that $A \cap \overline{B} \subseteq A - B$.
   Since $A - B \subseteq A \cap \overline{B}$ and $A \cap \overline{B} \subseteq A - B$, then $A - B = A \cap \overline{B}$.

   Brief Proof:
   Notice that $x \in A - B$ iff $x \in A$ and $x \notin B$ (by the definition of set difference)
   iff $x \in A$ and $x \in \overline{B}$ (by the definition of complement)
   iff $x \in A \cap \overline{B}$ (by the definition of intersection)
   Therefore membership in $A - B$ is equivalent to membership in $A \cap \overline{B}$, which implies that $A - B = A \cap \overline{B}$.

7. *Let a_1, a_2, …, a_n be real numbers. Prove that at least one of these numbers is greater or equal to the average of the numbers.*

   Proof (by contradiction):
   The average of the numbers is $A = (a_1 + a_2 + \ldots + a_n)/n$. Assume that none of these numbers is greater than or equal to A. That is, $a_i < A$ for all $i = 1, 2, \ldots n$. Thus $(a_1 + a_2 + \ldots + a_n) < An$. Solving for A, we get $A > (a_1 + a_2 + \ldots + a_n)/n = A$, which is a contradiction. Therefore at least one of the numbers is greater than or equal to the average.

8. *Prove that the square of a real number is non-negative.*

   Proof (by cases):
   Case 1: If $x > 0$, then we can multiply the inequality by x, and we get $x^2 > 0$, as required.
   Case 2: If $x = 0$, $x^2 = 0$, which is non-negative.
   Case 3: If $x < 0$, then we can write $y = -x$, where $y > 0$. Then

   $$x^2 = (-y)^2 = (-1y)^2 = (-1)^2 y^2 = y^2 > 0,$$

   the last step by Case 1.
   Since we have proved the statement for all cases, the statement is true.

9. *Show that* $\sum_{i=1}^{n}(a_i - a_{i-1}) = a_n - a_0$, *where* $a_0, a_1, \ldots, a_n$ *are real numbers.*

Proof:
We can see that

$$\begin{aligned}\sum_{i=1}^{n}(a_i - a_{i-1}) &= \left(\sum_{i=1}^{n} a_i\right) - \left(\sum_{i=1}^{n} a_{i-1}\right) \\ &= (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_0 + a_1 + a_2 + \cdots + a_{n-1}) \\ &= a_1 + a_2 + \cdots + a_{n-1} + a_n - a_0 - a_1 - a_2 - \cdots - a_{n-1} \\ &= (a_1 - a_1) + (a_2 - a_2) + \cdots + (a_{n-1} - a_{n-1}) + a_n - a_0 \\ &= a_n - a_0.\end{aligned}$$

10. *Prove that the sum of the first* n *odd integers is* n^2.

Proof:
The sum of the first n odd integers is

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n^2 + n - n = n^2.$$

11. *Prove that if* x *is a real number and* m *is an integer, then* $\lceil x + m \rceil = \lceil x \rceil + m$.

Proof:
Since x is a real number, there exists some integer N such that $N < x \leq N + 1$. Adding m to each part of the equation yields $N + m < x + m \leq N + 1 + m$. From these two sets of inequalities, we can see that $\lceil x + m \rceil = N + 1 + m = \lceil x \rceil + m$.

12. *Let* f *be a function from* B *to* C, *and* g *be a function from* A *to* B. *If both* f *and* g *are one-to-one, prove that* $f \circ g$ *is one-to-one.*

Direct Proof:
For any distinct elements $x, y \in A$, $g(x) \neq g(y)$, since g is one-to-one. Since f is also one-to-one, then $f(g(x)) \neq f(g(y))$ (which is the same as $((f \circ g)(x) \neq (f \circ g)(y))$. Therefore $f \circ g$ is one-to-one.

Proof by Contradiction:
Assume $f \circ g$ is not one-to-one. Then there exist distinct elements $x, y \in A$ such that $(f \circ g)(x) = (f \circ g)(y)$. This is equivalent $f(g(x)) = f(g(y))$. Since f is one-to-one, it must be the case that $g(x) = g(y)$. But $x \neq y$, and g is one-to-one, so $g(x) \neq g(y)$. This is a contradiction. Therefore $f \circ g$ is one-to-one.

13. *Let* a *and* b *be integers such that* $a|b$ *and* $b|a$. *Prove that either* $a = b$ *or* $a = -b$.

Proof:
If $a|b$, then we can write $b = ac$ for some integer c. Similarly, if $b|a$, then we can write $a = bd$ for some integer d. Then we can write $b = ac = (bd)c$. Dividing both sides by b (which is legal, since $b|a$ implies $b \neq 0$), we can see that $cd = 1$. Since c and d are integers, we know that either $c = d = 1$ or $c = d = -1$. In the first case, we have that $a = b$, and in the second case, we have that $a = -b$.