Proofs: Suggested Exercises

Some of the exercises are based on problems and examples from *Discrete Mathematics and its Applications, 4th Edition*, by Kenneth Rosen.

Logic

1. Given that the propositions $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ are all true, prove that t is true.
2. Given that "If you come home tonight, then I will make dinner," "If you do not come home tonight, then I will go play disc golf," and "If I go play disc golf, then I will have a good time," prove that "If I do not make dinner tonight, then I will have a good time." (Hint: Start by expressing the statements in terms of propositions like p, q, and r)
3. Given that $\forall x(A(x) \rightarrow B(x))$, and A(blah) are true, prove that B(blah) is true.
4. Given that "All lions are fierce" and "Some lions do not drink coffee," prove that "Some fierce creatures do not drink coffee." (Hint: This will involve quantifiers ($\forall$, $\exists$).)

Sets

Assume A, B, and C are sets.

1. Prove that $A \cap B \subseteq A \cup B$.
2. Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
3. Prove that $(A - B) - C \subseteq A - C$.
4. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Functions

1. Let x be real, and n be an integer. Prove that $x < n$ if and only if $\lfloor x \rfloor < n$. (Prove both ways.)
2. Let a and b be real numbers with $a \neq 0$. Show that the function $f(x) = a\,x + b$ is invertable.
3. Prove that if f and g are onto, then $f \circ g$ is also onto.
4. Prove or disprove: if a, b, and c are real numbers with $a \neq 0$, then the function $f(x) = a\,x^2 + b\,x + c$ is invertable.

Integers

1. Prove that the sum of two even integers is even.
2. Prove that the product of two rational numbers is rational.
3. Prove or disprove: Every positive integer can be written as the sum of the squares of two integers.
4. Prove that the product of a rational number and an irrational number is irrational.
5. Prove that if n is an integer, and $5n + 4$ is even, then n is even. Give a direct proof, an indirect proof, and a proof by contradiction.
6. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (Hint: Try a proof by cases.)
7. Prove that the square of an integer that is not divisible by 5 leaves a remainder of 1 or 4 when divided by 5. (Hint: Use a proof by cases. Case 1 should be "$n \equiv 1 \bmod 5$".)
8. Prove or disprove that $n^2 - 1$ is composite whenever n is a positive integer greater than or equal to 1.
9. Prove or disprove that $n^2 - 1$ is composite whenever n is a positive integer greater than or equal to 3.
10. Prove that if x, and y are integers, then $\gcd(x, y) \times \mathrm{lcm}(x, y) = x \times y$.