Sample Relation Proofs

1. *Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a,a) | a \in A\}$.*
   Proof: Assume that R is antisymmetric, but $R \cap R^{-1} \not\subseteq \Delta$. Then there are elements $a, b \in A$ with $a \neq b$ such that $(a,b) \in R \cap R^{-1}$. Thus, $(a,b) \in R$, and $(a,b) \in R^{-1}$. The latter implies that $(b,a) \in R$ by the definition of R^{-1}. But then we have $(a,b) \in R$, and $(b,a) \in R$, with $a \neq b$, contradicting that R is antisymmetric. Thus, $R \cap R^{-1} \subseteq \Delta$.
   Now assume that $R \cap R^{-1} \subseteq \Delta$, but R is not antisymmetric. Then there are elements $a, b \in A$ with $a \neq b$ such that $(a,b) \in R$ and $(b,a) \in R$. Then $(a,b) \in R^{-1}$ (since $(b,a) \in R$), so that $(a,b) \in R \cap R^{-1}$. Since $a \neq b$, this contradicts that $R \cap R^{-1} \subseteq \Delta$. Thus, R is antisymmetric.

2. *Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.*
   Proof: We need to show that R is reflexive, symmetric, and transitive.
   Reflexive: Since $ab = ba$ for all positive integers, $((a,b),(a,b)) \in R$ for all (a,b). Thus R is reflexive.
   Symmetric: Notice that if $ad = bc$, then $cb = da$ for all positive integers a, b, c, and d. Thus $((a,b),(c,d)) \in R$ implies that $((c,d),(a,b)) \in R$, so R is symmetric.
   Transitive: Assume that $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Then $ad = bc$ and $cf = de$. Solving the second for c, we get $c = de/f$, and plugging it into the first we get $ad = b(de/f)$. Multiplying both sides by f, and canceling the d on both sides yields $af = be$. Thus $((a,b),(e,f)) \in R$. Thus R is transitive.

3. *Let R be the relation on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that $(f,g) \in R$ if and only if f is $\Theta(g)$. Show that R is an equivalence relation.*
   Proof: We need to show that R is reflexive, symmetric, and transitive.
   Reflexive: Since $1 \cdot f(x) \leq f(x) \leq 1 \cdot f(x)$ for all $x \geq 1$, $f = \Theta(f)$, so R is reflexive.
   Symmetric: If $f = \Theta(g)$, then there exists positive constance C_1, C_2, and x_0 such that

   $$C_1 g(x) \leq f(x) \leq C_2 g(x) \text{ for all } x \geq x_0.$$

   This implies that

   $$g(x) \leq \frac{1}{C_1} f(x) \text{ and } g(x) \geq \frac{1}{C_2} f(x) \text{ for all } x \geq x_0,$$

   which is equivalent to

   $$\frac{1}{C_2} f(x) \leq g(x) \leq \frac{1}{C_1} f(x) \text{ for all } x \geq x_0.$$

   Thus $g = \Theta(f)$, and R is symmetric.
   Transitive: If $f = \Theta(g)$, then there exists positive constance C_1, C_2, and x_0 such that

   $$C_1 g(x) \leq f(x) \leq C_2 g(x) \text{ for all } x \geq x_0.$$

   Similarly if $g = \Theta(h)$, then there exists positive constance C_3, C_4, and x_1 such that

   $$C_3 h(x) \leq g(x) \leq C_4 h(x) \text{ for all } x \geq x_1.$$

   Then

   $$f(x) \geq C_1 g(x) \geq C_1(C_3 h(x)) \text{ for all } x \geq \max\{x_0, x_1\},$$

   and

   $$f(x) \leq C_2 g(x) \geq C_2(C_4 h(x)) \text{ for all } x \geq \max\{x_0, x_1\}$$

   Thus,

   $$C_1 C_3 h(x) \leq f(x) \leq C_2 C_4 h(x) \text{ for all } x \geq \max\{x_0, x_1\}.$$

   and $f = \Theta(h)$. Thus R is transitive.