Sections 1.3

Math 231

Hope College

The Dot Product

- Given vectors $\vec{\mathbf{x}} = \langle x_1, x_2, \ldots, x_n \rangle$ and $\vec{\mathbf{y}} = \langle y_1, y_2, \ldots, y_n \rangle$ in $\mathbb{R}^n$, we define the **dot product** of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ to be the scalar

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

- Note that the dot product of any two vectors is a *scalar*. A common mistake is forgetting to add the resulting products and leaving the final answer as a vector.
- **Theorem 1.20:** (Properties of the Dot Product)
  1. For all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$, $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = \vec{\mathbf{y}} \cdot \vec{\mathbf{x}}$.
  2. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $\vec{\mathbf{x}} \cdot \vec{\mathbf{x}} = \|\vec{\mathbf{x}}\|^2$.
  3. For all $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} \in \mathbb{R}^n$, $\vec{\mathbf{x}} \cdot (\vec{\mathbf{y}} + \vec{\mathbf{z}}) = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}} + \vec{\mathbf{x}} \cdot \vec{\mathbf{z}}$.
  4. For all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $(\alpha\vec{\mathbf{x}}) \cdot \vec{\mathbf{y}} = \vec{\mathbf{x}} \cdot (\alpha\vec{\mathbf{y}}) = \alpha(\vec{\mathbf{x}} \cdot \vec{\mathbf{y}})$.

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Cauchy-Schwarz and the Triangle Inequality

- **Theorem 1.22:**(Cauchy-Schwarz Inequality)

  For all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$,

  $$|\vec{\mathbf{x}} \cdot \vec{\mathbf{y}}| \leq \|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\| \, .$$

- **Theorem 1.23:** (The Triangle Inequality)

  For all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$,

  $$\|\vec{\mathbf{x}} + \vec{\mathbf{y}}\| \leq \|\vec{\mathbf{x}}\| + \|\vec{\mathbf{y}}\| \, .$$

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Angle and Orthogonality

- **Theorem 1.24:** (The Angle Between Two Vectors)

  If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and θ is the measure of the angle between them, measured so that $0 \leq \theta < \pi$, then

$$\theta = \cos^{-1} \frac{\vec{\mathbf{x}} \cdot \vec{\mathbf{y}}}{\|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\|}.$$

- Given two vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$ where $n \geq 4$, there is no predefined notion of the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$. Therefore, rather than proving a theorem such as 1.24 in the general case, we simply allow the formula above to *define* the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.
- Two vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$ are **orthogonal** if the angle between them is $\pi/2$. Equivalently, $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are orthogonal if $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = 0$.

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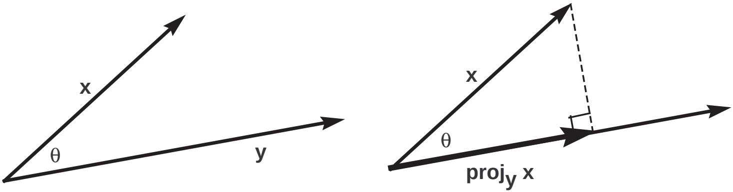
Projection

- Sometimes, it is useful to project one vector onto the direction defined by another vector.



This can be done using dot products:

Given $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ in $\mathbb{R}^n$, with $\vec{\mathbf{y}} \neq \vec{0}$, $\mathbf{proj}_{\vec{\mathbf{y}}}\, \vec{\mathbf{x}} = \dfrac{\vec{\mathbf{x}} \cdot \vec{\mathbf{y}}}{\vec{\mathbf{y}} \cdot \vec{\mathbf{y}}}\, \vec{\mathbf{y}}$.