Sample Decidable/Undecidable proofs

1. **Problem 4.3:** Let $\text{ALL}_{\text{DFA}} = \{ \langle A \rangle \mid A \text{ is a DFA that recognizes } \Sigma^* \}$. Show that $\text{ALL}_{\text{DFA}}$ is decidable.

   **Proof #1:** The following TM decides $\text{ALL}_{\text{DFA}}$:
   
   $S = "\text{On input } \langle A \rangle, \text{ where } A \text{ is a DFA:}\n   \begin{align*}
   &\text{1. Construct a DFA } B \text{ such that } L(B) \text{ is the complement of } L(A). \\
   &\text{2. Use the TM } T \text{ from Thm 4.4 (deciding } E_{\text{DFA}}) \text{ on input } <B> \\
   &\text{3. Accept if } T \text{ accepts, reject if } T \text{ rejects."} \\
   \text{Proof #2:** The following TM decides } \text{ALL}_{\text{DFA}}: 
   \begin{align*}
   S &= "\text{On input } \langle A \rangle, \text{ where } A \text{ is a DFA:}\n   \begin{align*}
   &\text{1. Construct a DFA } B \text{ such that } L(B) = \Sigma^*. \\
   &\text{2. Use the TM } F \text{ from Thm 4.5 (deciding } E_{\text{DFA}}) \text{ on input } <A,B> \\
   &\text{3. Accept if } F \text{ accepts, reject if } F \text{ rejects."} \\
   \end{align*}
   \end{align*}
   \]

2. **Problem 5.9:** Prove that $T = \{ \langle M \rangle \mid M \text{ is a Turing machine that accepts } w^R \text{ whenever it accepts } w \}$ is undecidable.

   **Proof:** Assume that $T$ is decidable. Then some TM $M$ decides $T$. We construct TM $M'$ as follows
   
   $M' = "\text{On input } x:\n   \begin{align*}
   &\text{1. If } x \neq 01 \text{ and } x \neq 10, \text{ reject.} \\
   &\text{2. If } x = 01, \text{ accept.} \\
   &\text{3. If } x = 10 \text{ simulate } M \text{ on } w. \\
   &\text{4. If } M \text{ accepts } w, \text{ accept. If } M \text{ halts and rejects, reject."} \\
   \]

   If $\langle M, w \rangle \in A_{\text{TM}}$ then $M$ accepts $w$ and $L(M') = \{01, 10\}$, so $M' \in T$.
   On the other hand, if $\langle M, w \rangle \notin A_{\text{TM}}$ then $L(M') = \{01\}$, so $M' \notin T$.
   Thus, $\langle M, w \rangle \in A_{\text{TM}}$ iff $M' \in T$, so $A_{\text{TM}} \leq_m T$. Therefore $A_{\text{TM}}$ is decidable. Since $A_{\text{TM}}$ is undecidable, it must be the case that our assumption that $T$ is decidable is false, so $T$ is undecidable.

3. **Problem 5.22:** Prove that $A$ is Turing-recognizable iff $A \leq_m A_{\text{TM}}$.

   **Proof:**
   
   $\Leftarrow$ Assume $A \leq_m A_{\text{TM}}$. According to Thm 4.11a (not in the book as theorem, but stated on page 174), $A_{\text{TM}}$ is Turing-recognizable. Thus, by Thm 5.28, $A$ is Turing-recognizable.
   
   $\Rightarrow$ Assume that $A$ is Turing-recognizable. Then there is some TM $M$ which recognizes $A$. Let $f(x) = \langle M, x \rangle$. First notice that $f$ is easily computable. Second, notice that $x \in A$ iff $M$ accepts $x$ iff $\langle M, x \rangle \in A_{\text{TM}}$, so $f$ is a mapping reduction. Thus, $A \leq_m A_{\text{TM}}$. 