Sample Decidable/Undecidable proofs

1. **Problem 4.3:** Let $ALL_{\text{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA that recognizes } \Sigma^*\}$. Show that ALL_{DFA} is decidable.

   Proof #1: The following TM decides ALL_{DFA}:
   S = "On input $\langle A\rangle$, where A is a DFA:
      1. Construct a DFA B such that L(B) is the complement of L(A).
      2. Use the TM T from Thm 4.4 (deciding E_{DFA}) on input $\langle B\rangle$
      3. Accept if T accepts, reject if T rejects."

   Proof #2: The following TM decides ALL_{DFA}:
   S = "On input $\langle A\rangle$, where A is a DFA:
      1. Construct a DFA B such that L(B) = Σ^*.
      2. Use the TM F from Thm 4.5 (deciding EQ_{DFA}) on input $\langle A,B\rangle$
      3. Accept if F accepts, reject if F rejects."

2. **Problem 5.9:** Prove that $T = \{\langle M\rangle \mid M \text{ is a Turing machine that accepts } w^R \text{ whenever it accepts } w\}$ is undecidable.
   Proof: Assume that T is decidable. Then some TM M decides T. We construct TM M' as follows
   M' = "On input x:
      1. If $x \neq 01$ and $x \neq 10$, reject.
      2. If $x = 01$, accept.
      3. If $x = 10$ simulate M on w.
      4. If M accepts w, accept. If M halts and rejects, reject."

   If $\langle M,w\rangle \in A_{\text{TM}}$ then M accepts w and $L(M')=\{01,10\}$, so $M' \in T$.
   On the other hand, if $\langle M,w\rangle \notin A_{\text{TM}}$ then $L(M')=\{01\}$, so $M' \notin T$.
   Thus, $\langle M,w\rangle \in A_{\text{TM}}$ iff $M' \in T$, so $A_{\text{TM}} \leq_m T$. Therefore A_{TM} is decidable. Since A_{TM} is undecidable, it must be the case that our assumption that T is decidable is false, so T is undecidable.

3. **Problem 5.22:** Prove that A is Turing-recognizable iff $A \leq_m A_{\text{TM}}$.
   Proof :

   $\Leftarrow$ Assume $A \leq_m A_{\text{TM}}$. According to Thm 4.11a (not in the book as theorem, but stated on page 174), A_{TM} is Turing-recognizable. Thus, by Thm 5.28, A is Turing-recognizable.

   $\Rightarrow$ Assume that A is Turing-recognizable. Then there is some TM M which recognizes A. Let $f(x) = \langle M,x\rangle$. First notice that f is easily computable. Second, notice that $x \in A$ iff M accepts x iff $\langle M,x\rangle \in A_{\text{TM}}$, so f is a mapping reduction. Thus, $A \leq_m A_{\text{TM}}$.