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Two-pebbling and odd-two-pebbling are not equivalent

Charles A. Cusack [a,b,*], Airat Bekmetjev [b], Mark Powers [a]

[a] *Department of Computer Science, Hope College, 27 Graves Place, Holland, MI 49422, United States*
[b] *Department of Mathematics, Hope College, 27 Graves Place, Holland, MI 49422, United States*

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ABSTRACT

Let G be a connected graph. A configuration of pebbles assigns a nonnegative integer number of pebbles to each vertex of G. A move consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. A configuration is solvable if any vertex can get at least one pebble through a sequence of moves. The pebbling number of G, denoted $\pi(G)$, is the smallest integer such that any configuration of $\pi(G)$ pebbles on G is solvable. A graph has the two-pebbling property if after placing more than $2\pi(G) - q$ pebbles on G, where q is the number of vertices with pebbles, there is a sequence of moves so that at least two pebbles can be placed on any vertex. A graph has the odd-two-pebbling property if after placing more than $2\pi(G)-r$ pebbles on G, where r is the number of vertices with an odd number of pebbles, there is a sequence of moves so that at least two pebbles can be placed on any vertex. In this paper, we prove that the two-pebbling and odd-two-pebbling properties are not equivalent.

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1. Introduction

Let G be a connected graph. A *configuration* assigns a nonnegative number of pebbles to the vertices of G. For a configuration C, we define $C(v)$ to be the number of pebbles on vertex v, and if U is a subset of vertices of G, then $C(U)$ is the total number of pebbles on the vertices in U. A *pebbling move* (or just *move*) removes two pebbles from one vertex and places one pebble on an adjacent vertex. A vertex v is *reachable* under some configuration if it is possible to move a pebble to v through a sequence of pebbling moves. A configuration is *solvable* if all vertices are reachable. The *pebbling number rooted at a vertex* v in G, $\pi(G, v)$, is defined as the smallest number of pebbles so that for any configuration of $\pi(G, v)$ pebbles, v is reachable. The *pebbling number* of a graph is $\pi(G) = \max_{v\in V(G)} (\pi(G, v))$.

A graph G has the *two-pebbling property* if for every configuration of more than $2\pi(G) - q$ pebbles, where q is the number of vertices with pebbles, it is possible to move 2 pebbles to any vertex. A *violating configuration* for a vertex v of G is any configuration of more than $2\pi(G) - q$ pebbles such that two pebbles cannot be moved to v. A graph that does not have the two-pebbling property is called a *Lemke graph*.

The two-pebbling property was introduced by Chung [1]. Most graphs have the two-pebbling property [8]. In fact, only a handful of families of Lemke graphs have been found [2–4,9,10]. Graham's Conjecture states for any two graphs G and H, $\pi(G\square H) \leq \pi(G)\pi(H)$, where $G\square H$ is the Cartesian product of G and H [1]. Graham's conjecture has been studied by numerous researchers, and many results that affirm the conjecture rely on the two-pebbling property [1,5–7,9,10].

A graph G has the *odd-two-pebbling property* if for every configuration of more than $2\pi(G) - r$ pebbles, where r is the number of vertices with an odd number of pebbles, it is possible to move 2 pebbles to any vertex [10]. Note that any graph

* Corresponding author at: Department of Computer Science, Hope College, 27 Graves Place, Holland, MI 49422, United States.
E-mail addresses: cusack@hope.edu (C.A. Cusack), bekmetjev@hope.edu (A. Bekmetjev).

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which has the two-pebbling property also has the odd-two-pebbling property. All Lemke graphs found to date also do not have the odd-two-pebbling property. This is true even of more recent Lemke graphs [2,3]. Wang conjectured that two-pebbling and odd-two-pebbling are equivalent [10]. We present a graph that has the odd-two-pebbling property but does not have the two-pebbling property, proving that the properties are not equivalent.

2. General results

The following is a somewhat obvious but powerful tool in analyzing Lemke graphs.

Theorem 1. *Let C be a violating configuration on graph G for root r with $2\pi(G)-q+k$ pebbles, where $k \geq 1$. Then it is impossible to place a pebble on r using less than $\pi(G)-q+k+1$ pebbles.*

Proof. If $\pi(G)-q+k$ pebbles are used to place one pebble on r, $\pi(G)$ pebbles are left on G so a second pebble can be moved to r.

In our arguments related to the two-pebbling property, we will often state that the root can be reached using $\pi(G)-q+1$ pebbles and leave implicit the fact that a second pebble can be moved to the root by Theorem 1, implying that the given configuration is not a violating configuration for the given root.

Lemma 2. *Let G be a graph with n vertices and let C be a violating configuration for root r. Then $q < n$ and $C(r) = 0$.*

Proof. If $q = n$, then there are at least $2\pi(G) - n + 1 \geq 2n - n + 1 = n + 1$ pebbles on n vertices. Since every vertex has at least one pebble and at least one vertex has at least two pebbles, a second pebble can be moved to any vertex. Clearly $C(r) < 2$. If $C(r) = 1$, then there are at least $2\pi(G) - q + 1 - 1 = \pi(G) + (\pi(G) - q) \geq \pi(G)$ other pebbles on the graph and a second pebble can be moved to r.

Lemma 3. *Let G be a Hamiltonian graph with n vertices, C a configuration with $p \geq n + 2$ pebbles on $q = n - 1$ vertices. Then two pebbles can be moved to any vertex in G.*

Proof. Since some vertex has at least two pebbles, any vertex that already has a pebble can get a second pebble by pebbling along the Hamiltonian cycle. Let r be the vertex without a pebble. Since $p = n + 2$, either two vertices, u and v, have at least two pebbles or some vertex u has 4 pebbles. In the first case, a pebble can be moved to r from each of u and v along two disjoint paths that are part of the Hamiltonian cycle. Similarly, if some vertex has 4 pebbles, two pebbles can be moved to r from u by following two disjoint paths along the Hamiltonian cycle.

Lemma 4. *Let C be a violating configuration, u be a vertex with $C(u) \geq 3$, and assume $C(v) = 0$ for some neighbor of v of u. Create configuration C' from C by moving one pebble from u to v. Then C' is a violating configuration.*

Proof. Let C be a violating configuration for some root r with p pebbles on $q - 1$ vertices such that $C(u) \geq 3$, and let v be a neighbor of u with $C(v) = 0$. Since C is a violating configuration, $p + q - 1 > 2\pi(G)$. Then C' has $p - 1$ pebbles on q vertices. Since $p - 1 + q > 2\pi(G)$ and r is still not reachable with two pebbles, C' is clearly a violating configuration.

Corollary 5. *Let G be a graph that has no violating configurations with pebbles on q vertices and let C be a violating configuration with pebbles on $q - 1$ vertices. If $C(u) \geq 3$, then for each neighbor v of u, $C(v) \geq 1$. Equivalently, if $C(v) = 0$, then $C(u) \leq 2$ for each neighbor u of v.*

The following lemma is straightforward.

Lemma 6. *Let P_n be a path on n vertices, K_3 be a clique on 3 vertices with vertex set $V(K_3) = \{v_1, v_2, v_3\}$, and let C be a pebbling configuration.*

1. *If $n \leq 4$ and $C(P_n) \geq n + 1$, then at least two pebbles can be moved to one of its endpoints.*
2. *If $C(K_3) \geq 4$, then it is possible to move 2 pebbles to at least two of its vertices.*
3. *If $C(K_3) \geq 5$, then 2 pebbles can be moved to any of its vertices.*
4. *If $C(K_3) \geq 6$, then 4 pebbles can be moved to one of its vertices. Further, if $C(v_1) + C(v_2) \geq 6$ then 2 pebbles can be placed on both v_1 and v_2 simultaneously.*
5. *If $C(K_3) = 7$ and 4 pebbles cannot be moved to v_1 or v_2, then $C(v_3) = 5$ and $C(v_1) = C(v_2) = 1$ or $C(v_3) = 7$ and $C(v_1) = C(v_2) = 0$.*
6. *If $C(K_3) = 8$ and 4 pebbles cannot be moved to v_1 then $C(v_1) = 0$ and $C(v_2)$ and $C(v_3)$ are both odd.*
7. *If $C(K_3) \geq 9$, then 4 pebbles can be moved to any of its vertices.*
8. *If $C(K_3) \geq 14$ and each vertex has at least one pebble, then 4 pebbles can be moved to any two of its vertices simultaneously.*

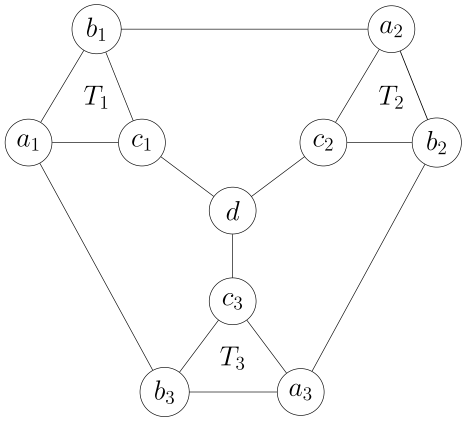

Fig. 1. The new Lemke graph, H.

3. The new Lemke graph

When the algorithm from [2] to determine whether or not a graph has the two-pebbling property was run on all ten-vertex graphs with diameter three, several new Lemke graphs were discovered with a very interesting property: all of the violating configurations have at least one vertex with an even number of pebbles. In other words, these are Lemke graphs that have the odd-two-pebbling-property. Since this was an unexpected result, it seemed prudent to verify it. The goal of this paper is to prove that one of these graphs, H (see Fig. 1), does not have the two-pebbling property but does have the odd-two-pebbling-property, proving that these two properties are not equivalent. We will proceed by showing that $\pi(H) = 10$ and then prove that H has exactly 6 violating configurations, none of which satisfy the conditions of the odd-two-pebbling property.

Let T_i be subgraph induced by vertices $\{a_i, b_i, c_i\}$ for $i \in \{1, 2, 3\}$. Let C be a configuration on H. Let $p_i = C(T_i)$ and q_i be the number of vertices on T_i with pebbles. Finally, let $\alpha_i = C(a_i)$, $\beta_i = C(b_i)$, $\gamma_i = C(c_i)$, and $\delta = C(d)$.

If moves are made on a configuration C, the result is a new configuration that is usually given a new name (e.g. C'). To simplify the notation in proofs, we will often continue to call the configuration C and use definitions from above even after moves have been made.

4. Pebbling number

In this section we show that $\pi(H) = 10$. Clearly $\pi(H, v) \geq 10$ for all v. Due to the symmetry of H, we prove that $\pi(H, d) = \pi(H, a_3) = \pi(H, c_3) = 10$ and the result follows.

Theorem 7. $\pi(H, d) = 10$.

Proof. Let C be a configuration with 10 pebbles such that d is unreachable. By Lemma 6.3, $p_i \leq 4$ for $i \in \{1, 2, 3\}$. Without loss of generality, we may assume that $p_1 = 4$ and that $\alpha_1 = 3$ and $\beta_1 = 1$ (due to symmetry and unreachability of d). Further, it is impossible to move a pebble from T_2 or T_3 to T_1. If $p_2 = 4$ or $p_3 = 4$, Lemma 6.2 implies that a pebble can be moved to either d or to T_1, so $p_2 = p_3 = 3$. The pebbles on T_2 do not allow a pebble to be moved to either d or b_1. Thus, either $\alpha_2 = \beta_2 = \gamma_2 = 1$ or $\beta_2 = 3$ and $\alpha_2 = \gamma_2 = 0$. In the former case, d is clearly reachable along the path (a_1, b_1, a_2, c_2, d), so $\beta_2 = 3$ and $\alpha_2 = \gamma_2 = 0$. A similar argument shows that $\alpha_3 = 3$ and $\beta_3 = \gamma_3 = 0$. But then a pebble can be moved from a_3 to b_2 so that b_2 has four pebbles and d is reachable. Therefore, $\pi(H, d) = 10$.

Theorem 8. $\pi(H, c_3) = 10$.

Proof. Let C be a configuration of 10 pebbles on H. Without loss of generality, assume $p_1 \geq p_2$. If any of $\{b_3, a_3, d\}$ has two or more pebbles c_3 is reachable, so assume otherwise. There are 4 cases to consider.

Case 1: All three of $\{b_3, a_3, d\}$ have one pebble. Then $p_1 \geq 4$ and c_3 is reachable by Lemma 6.2.

Case 2: Two of $\{b_3, a_3, d\}$ have one pebble. If $p_1 \geq 5$, then both b_3 and d are reachable from T_1 by Lemma 6.3. Since at least one of these has a pebble, c_3 is reachable. Otherwise, $p_1 = p_2 = 4$. If $\delta = 0$, then $\beta_3 = \alpha_3 = 1$ and c_3 is reachable unless $\alpha_1 = \beta_2 = 0$. In this case, a pebble can be moved to d from both T_1 and T_2, so c_3 is reachable. If $\delta = 1$, then without loss of generality, $\alpha_3 = 0$ and $\beta_3 = 1$, and Lemma 6.2 implies that a pebble can be moved to either d or b_3 from T_1, making c_3 reachable.

Case 3: There is one pebble on $\{b_3, a_3, d\}$. In this case, $p_1 \geq 5$. If $\beta_3 = 1$ or $\delta = 1$, c_3 is reachable by Lemma 6.3, so assume $\alpha_3 = 1$. This implies that $\beta_2 \leq 1$. If $p_1 \geq 8$, clearly c_3 is reachable from T_1. This leaves 3 subcases.

Case 3.1: $p_1 = 5$. Then by Lemma 6.3, a pebble can be moved from T_1 to T_2, putting 5 pebbles on T_2, which allows for a move to a_3, making c_3 reachable.

Case 3.2: $p_1 = 6$. Then $p_2 = 3$. If $\beta_2 = 1$, then a_2 either has a pebble or can receive one from c_2, so a pebble can be moved from T_1 along the path (a_2, b_2, a_3, c_3). If $\beta_2 = 0$, there are two cases to consider. If $\alpha_2 = 3$, a move can be made from T_1 to a_2, making c_3 reachable through a_3. Otherwise, $\alpha_2 \leq 2$ and $\gamma_2 \geq 1$, so d can be reached from both T_1 and T_2 and c_3 is reachable.

Case 3.3: $p_1 = 7$. Then $p_2 = 2$. By Lemma 6.5, if c_3 is not reachable from T_1, then either $\beta_1 = 5$ and $\alpha_1 = \gamma_1 = 1$, or $\beta_1 = 7$ and $\alpha_1 = \gamma_1 = 0$. If d, b_1, or a_3 is reachable from T_2, the configuration is solvable. Thus, two vertices in T_2 have one pebble. If $\beta_2 = 0$, then one pebble from T_1 can be moved through T_2 to d, leaving 5 pebbles on T_1, allowing another pebble to reach d. Otherwise, $\beta_2 = 1$ and we move two pebbles from b_1 to a_2 and then pebble along the path (a_2, b_2, a_3, c_3).

Case 4: There are no pebbles on $\{b_3, a_3, d\}$. If $p_1 = p_2 = 5$, then 2 pebbles can be moved to d and one to c_3. If $p_1 \geq 8$, clearly c_3 is reachable. This leaves two cases.

Case 4.1: $p_1 = 6$. Then $p_2 = 4$. If d is reachable from T_2, then c_3 is reachable. Otherwise, either $\beta_2 = 3$ and $\alpha_2 = 1$ or $\beta_2 = 1$ and $\alpha_2 = 3$. If $\beta_2 = 3$, then move a pebble from b_2 to a_3. By Lemma 6.3, two pebbles can be moved to b_1 and then a pebble can be moved along the path $(b_1, a_2, b_2, a_3, c_3)$. If $\alpha_2 = 3$, a move from a_2 to b_1 would place 7 pebbles on T_1. If that configuration is unsolvable for c_3, Lemma 6.5 implies that in the initial configuration either $\alpha_1 = \gamma_1 = 1$ and $\beta_1 = 4$, or $\beta_1 = 6$. In either case, moving a pebble from b_1 to a_2 instead allows pebbling to d from both T_1 and T_2, making c_3 reachable.

Case 4.2: $p_1 = 7$. Then $p_2 = 3$. If c_3 is unreachable from T_1, then Lemma 6.5 implies that $\beta_1 = 5$ and $\alpha_1 = \gamma_1 = 1$, or $\beta_1 = 7$ and $\alpha_1 = \gamma_1 = 0$. If a pebble can be moved from T_2 to T_1, then c_3 is reachable since T_1 now has 8 pebbles. If a pebble can be moved from T_2 to d, c_3 is also reachable. Otherwise, $\beta_2 = 3$ or $\alpha_2 = \beta_2 = \gamma_2 = 1$. If $\beta_2 = 3$, then 2 pebbles can be moved from b_1 to a_2, one pebble from b_2 to a_3, and then one pebble can be moved along the path $(b_1, a_2, b_2, a_3, c_3)$. If $\alpha_2 = \beta_2 = \gamma_2 = 1$, move along the path (b_1, a_2, c_2, d) and the remaining pebbles on T_1 allow a second pebble to be moved to d, so c_3 is reachable.

Let H_1 be the subgraph induced by the set of vertices $\{a_1, b_1, c_1, b_3\}$ and $H_2 = H \backslash H_1$. We will prove several results that will be used in the next theorem.

Lemma 9. *Let C be a configuration on H.*

1. *If $p_2 = 4$, then one pebble can be moved to a_3 unless $\beta_2 = 0$ and α_2 and γ_2 are both odd.*
2. *If $C(H_2) = 6$, then a pebble can be moved to a_3 unless $\delta = \alpha_2 = 3$.*
3. *If $C(H_2) \geq 7$, then a pebble can be moved to a_3.*

Proof. The proof of statement 1 is straightforward.

For statement 2, let $C(H_2) = 6$ and assume $\alpha_3 = 0$. By Lemma 6.3, a_3 is reachable if $p_2 \geq 5$. Thus, assume $p_1 \leq 4$ and therefore $\gamma_3 + \delta \geq 2$.

If $\gamma_3 + \delta = 2$, then $p_2 = 4$. By statement 1, we can assume $\alpha_2 = 1$ and $\gamma_2 = 3$ or $\alpha_2 = 3$ and $\gamma_2 = 1$. If $\gamma_3 = \delta = 1$, a_3 is clearly reachable. Otherwise, $\delta = 2$, and we can get 4 pebbles to either a_2 or c_2, thus allowing a pebble to be moved to a_3.

If $\gamma_3 + \delta = 3$, then $\delta = 3$ and $\gamma_3 = 0$ or we can clearly reach a_3. In this case, $p_2 = 3$ and unless $\alpha_2 = 3$ (the exception in the statement), either 2 pebbles can be moved to b_2 or one more pebble to d, and a_3 is reachable. Finally, a_3 is clearly reachable if $\gamma_3 + \delta \geq 4$.

For statement 3, if $C(H_2) = 7$, it is possible to remove one pebble from C and avoid the configuration with $\delta = \alpha_2 = 3$. By statement 2, a_3 is reachable.

Theorem 10. *$\pi(H, a_3) = 10$.*

Proof. Let C be a configuration of 10 pebbles on H and assume $\alpha_3 = 0$. By Lemma 9.3, a_2 is reachable if $C(H_2) \geq 7$. This leaves 6 cases.

Case 1: $C(H_2) = 6$. By Lemma 9.2, a_3 is reachable unless $\delta = \alpha_2 = 3$. In this case, a pebble can be moved from a_2 to b_1 so that the path $\{b_3, a_1, b_1, c_1\}$ has 5 pebbles. By Lemma 6.1, two pebbles can be moved to either c_1, in which case a fourth pebble can be added to d, or to b_3. In both cases, a_3 can be reached.

Case 2: $C(H_2) = 5$. Then $C(H_1) = 5$ and by Lemma 6.1 at least 2 pebbles can be moved to b_3 or c_1 (by considering the path $\{b_3, a_1, b_1, c_1\}$), and at least 2 pebbles can be moved to b_3 or b_1 (by considering the path $\{b_3, a_1, c_1, b_1\}$). If two pebbles can be moved to b_3, then a_3 is reachable, so we can assume that 2 pebbles can be moved to either c_1 or b_1. If $\alpha_2 = 3$, move a pebble from b_1 to a_2. Otherwise, move a pebble from c_1 to d. In either case, H_2 now has 6 pebbles and $\alpha_2 \neq 3$, so a_3 is reachable by Lemma 9.2.

For the remaining cases, since $C(H_1) \geq 6$, we can assume $\beta_3 = 0$ since otherwise a_3 is reachable. Thus, $p_1 = C(H_1) \geq 6$. We will assume that a_3 is not reachable from T_1, so Lemma 6.4 implies that 4 pebbles can be moved to either b_1 or c_1. This implies that two pebbles can be moved to either d or a_2 from T_1.

Case 3: $C(H_2) = 4$. If $\alpha_2 = 3$, move a pebble from T_1 to a_2 and a_3 is reachable. Similarly if $\delta = 3$. Otherwise, move two pebbles to either d or a_2 from T_1 so that $C(H_2) = 6$. Since it is not that case that both $\alpha_2 = 3$ and $\delta = 3$, a_3 is reachable by Lemma 9.2.

For the remaining cases $C(H_2) \leq 3$, so $p_1 \geq 7$. We will assume that 4 pebbles cannot be moved to a_1 since otherwise a_3 is reachable. Thus, $\alpha_1 \leq 1$, so $\beta_1 + \gamma_1 \geq 6$, and by Lemma 6.4, two pebbles can be placed on b_1 and c_1 simultaneously.

Case 4: $C(H_2) = 3$, so $p_1 = 7$.

Case 4.1: $p_2 = 3$. If 4 pebbles can be moved to b_1 from T_1, then 2 pebbles can be moved from T_1 to T_2 and by Lemma 6.3, a_3 is reachable. If 4 pebbles cannot be moved to either a_1 or b_1, then by Lemma 6.5, $\gamma_1 \geq 5$, so a pebble can be added to either a_2 or c_2. Since either $\beta_2 = 1$ or the parity of α_2 and γ_2 differ, it is possible to move to either a_2 or c_2 so that a_3 is reachable by Lemma 9.1.

Case 4.2: $p_2 = 2$. Then $\delta = 1$ or $\gamma_3 = 1$. If 4 pebbles can be moved to b_1 from T_1, then by Lemma 9.1, we can assume $\gamma_2 = \alpha_2 = 1$ and $\beta_2 = 0$ since otherwise a_3 is reachable. Move 2 pebbles to both b_1 and c_1. If $\delta = 1$, move a pebble along the paths (b_1, a_2, b_2) and (c_1, d, c_2, b_2) so that b_2 has two pebbles. If $\gamma_3 = 1$, move a pebble from c_1 to d and along the path (b_1, a_2, c_2, d). In either case, a_3 can be reached.

If 4 pebbles cannot be moved to either a_1 or b_1, by Lemma 6.5, either $\gamma_1 = 5$ and $\beta_1 = 1$ or $\gamma_1 = 7$ and $\beta_1 = 0$. Since 2 pebbles can be moved to d, a_3 is reachable if $\gamma_3 = 1$, so assume $\delta = 1$. If $\gamma_1 = 7$ then we can move 3 more pebbles to d. Otherwise, $\gamma_1 = 5$ and $\alpha_1 = \beta_1 = 1$. Since $p_2 = 2$, there are four possibilities. If $\beta_2 = 1$, then either $\alpha_2 = 1$ or $\gamma_2 = 1$ and a second pebble can be added to either a_2 or c_2 and then to b_2. If $\gamma_2 = 2$ or $\alpha_2 = 2$ then 4 pebbles can be moved to a_1. If $\gamma_2 = \alpha_2 = 1$ then move a pebble along the paths (c_1, b_1, a_2, b_2) and (c_1, d, c_2, b_2) so b_2 has two pebbles. In all cases, a_3 is reachable.

Case 4.3: $p_2 \leq 1$. Then $\delta + \gamma_3 \geq 2$. If $\gamma_3 \geq 2$, $\delta \geq 3$, or both $\delta \geq 1$ and $\gamma_3 \geq 1$, then a_3 is clearly reachable. Thus, $\delta = 2$ and $\gamma_3 = 0$. Then $p_1 = 1$ and either 4 pebbles can be moved to c_1 or, by Lemma 6.5, either $\beta_1 = 5$ and $\gamma_1 = \alpha_1 = 1$ or $\beta_1 = 7$ and $\alpha_1 = \gamma_1 = 0$. If $\beta_2 = 1$ or $\gamma_2 = 1$ then 2 pebbles can be moved to b_2, so $\alpha_2 = 1$. If $\beta_1 = 7$, we can move 3 more pebbles to a_2. Otherwise, $\beta_1 = 5$ and $\alpha_1 = \gamma_1 = 1$, so a second pebble can be moved to a_1 from d and 2 more pebbles can be moved to a_1 from b_1. In any case, a_3 is reachable.

Case 5: $C(H_2) = 2$. Then $p_1 = 8$ and if a_3 is not reachable from T_1, then Lemma 6.6 implies that $\alpha_1 = 0$ and $\beta_1 + \gamma_1 = 8$, where both are odd. No matter how these pebbles are placed, both d and a_2 are reachable with 2 pebbles from T_1, and one of them can receive 3 pebbles. Thus, a_3 is reachable if b_2 or c_3 has one pebble, d, c_2, or a_2 has two pebbles, or both d and a_2 have one pebble. Thus, we can assume either $\delta = \gamma_2 = 1$ or $\gamma_2 = \alpha_2 = 1$, and it is straightforward to verify that a_3 can be reached from any of the eight configurations on T_1.

Case 6: $C(H_2) \leq 1$. Then $p_1 = 9$ and the a_3 is reachable by Lemma 6.7.

5. Two-pebbling property

Lemma 11. *Let C be a violating configuration on H with pebbles on q vertices. Then $4 \leq q \leq 7$.*

Proof. Let C be a configuration of $p = 21 - q$ pebbles on q vertices of H. If $q = 1$, $p = 20 = 2\pi(H)$ and 2 pebbles can be moved to any vertex.

If $q = 2$, $p = 19$, and some vertex u has at least ten pebbles. Since the diameter of H is 3, moving from u to any other vertex uses at most 8 pebbles, leaving at least 11 pebbles, enough to move a second pebble that vertex.

If $q = 3$, $p = 18$. Each of the three vertices with a pebble has at most 7 pebbles since otherwise one pebble can be placed on any other vertex leaving $\pi(H)$ pebbles on the graph, so a second pebble can be moved to that vertex. Thus each of the three vertices with pebbles, u, v, and w, has between 4 and 7 pebbles. No matter which vertices u, v, and w are, every vertex is within distance two of one of them. Thus, one pebble can be moved to any root using 4 pebbles, leaving 14 pebbles to move a second pebble.

By Lemma 2, $q \leq 9$ and r has no pebbles. If $q = 9$, $p = 12$ and since H is Hamiltonian, the result follows from Lemma 3.

If $q = 8$, r and some other vertex v have no pebbles. Since $p = 13$ and $H \setminus \{v\}$ is Hamiltonian, the result follows from Lemma 3.

Theorem 12. *There are no configurations on H that violate the two-pebbling-property with d as the root.*

Proof. Let C be a violating configuration for vertex d with $21 - q$ pebbles on q vertices. By Lemma 11, we only need to consider $4 \leq q \leq 7$. In all of these cases, $p \geq 14$. Therefore, $p_i \geq 5$ for some i. No matter how those pebbles are placed on T_i, d is reachable using only 4 pebbles, and the result follows from Theorem 1.

Lemma 13. *Let C be a configuration on H with $\alpha_1 \geq 1$.*

1. *If $\alpha_1 + \gamma_1 \geq 7$, $\alpha_1 + \beta_1 \geq 7$, or $\alpha_1 + \beta_1 + \gamma_1 \geq 8$, then a pebble can be moved to a_3.*
2. *If $p_1 = 14$ and either $\beta_1 \geq 6$ and $a_2 \geq 1$ or $\gamma_1 \geq 6$ and $d \geq 1$, two pebbles can be moved to a_3.*

Proof. The proof of statement 1 is straightforward. For the second statement, move 3 pebbles from b_1 to a_2 (or from c_1 to d) and then a pebble can be moved from a_2 (or d) to a_3. Since $p_1 = 8$ now, the result follows from statement 1.

Lemma 14. *Let C be a violating configuration on H with root a_3 with pebbles on $q \leq 7$ vertices such that there are no violating configurations on $q + 1$ vertices. Then $p_2 \leq 3$.*

Proof. If $p_1 \geq 5$, Lemma 6.3 implies that a_3 is reachable from T_2. It is not too difficult to see that it requires at most 4 of the 5 pebbles, leaving at least 10 pebbles on H, allowing a second pebble to reach a_3. When $p_2 = 4$, each configuration either allows a_3 to be reachable with at most 4 pebbles or violates Corollary 5.

Theorem 15. *H has no violating configurations with root a_3.*

Proof. Let C be a violating configuration with $21-q$ pebbles on q vertices. By Lemma 11, we only need to consider $4 \leq q \leq 7$. In all of these cases, $p \geq 14$. By Lemma 14, $p_2 \leq 3$. Also, $\gamma_3 + \delta \leq 3$ and $\beta_3 \leq 1$ by Theorem 1. This implies that $p_1 \geq 7$. Corollary 5 implies that $q_1 = 3$. Using Theorem 1 again, $\beta_3 = 0$, and thus $p_1 \geq 8$. Once again, Corollary 5 implies that $1 \leq \alpha_1 \leq 2$ (a fact we use often when applying Lemma 13.1), so $\beta_1 + \gamma_1 \geq 6$. By Theorem 1, if $\beta_1 \geq 2$, at least one of α_2 and β_2 is zero, and if $\gamma_1 \geq 2$, at least one of δ and γ_3 is zero. Since $\beta_1 + \gamma_1 \geq 6$, it follows that at least one of α_2, β_2, δ, and γ_3 is zero.

Case 1: $q = 7$. Then $p = 14$. Since $\alpha_3 = \beta_3 = 0$, exactly one other vertex has no pebbles. Thus, either $\delta = \gamma_3 = 1$ or $\alpha_2 = \beta_2 = 1$. In either case, $\gamma_2 = 1$.

Case 1.1: $\delta = \gamma_3 = 1$. Then $\gamma_1 = 1$, so $\beta_1 \geq 5$, and Corollary 5 implies that $1 \leq \alpha_2 \leq 2$, so that $\beta_2 = 0$. If $\alpha_2 = 2$, move a pebble along (a_2, c_2, d, c_3, a_3) and Lemma 6.7 implies that a_3 is reachable with a second pebble since $p_1 = 9$. If $\alpha_2 = 1$, then $\alpha_1 + \beta_1 = 9$. Move along (b_1, c_1, d, c_3, a_3) leaving $\alpha_1 + \beta_1 \geq 7$, so Lemma 13.1 applies.

Case 1.2: $\alpha_2 = \beta_2 = 1$. Then $\beta_1 = 1$, and $\delta + \gamma_3 = 1$, so $\alpha_1 + \gamma_1 = 9$. Pebble along $(c_1, b_1, a_2, b_2, a_3)$ leaving $\alpha_1 + \gamma_1 \geq 7$, so Lemma 13.1 applies.

Case 2: $q = 6$. Then $p = 15$. Since $p_2 \leq 3$ and $\delta + \gamma_3 \leq 3$, then $p_1 \geq 9$. Theorem 1 implies that either $\delta = 0$ or $\gamma_3 = 0$, and either $\alpha_2 = 0$ or $\beta_2 = 0$, and all other vertices have at least one pebble. This gives us four cases.

Case 2.1: $\delta = \alpha_2 = 0$. Corollary 5 implies that a_1, b_1, and c_1 each have at most two pebbles, contradicting the fact that $p_1 \geq 9$.

Case 2.2: $\delta = \beta_2 = 0$. Then $\gamma_3 = 1$, $\gamma_1 \leq 2$, and $\gamma_2 + \alpha_2 \leq 3$, so $p_1 \geq 11$ and $\beta_1 \geq 7$. If $\gamma_1 = 2$, move a pebble from c_1 to d and then move a pebble along $(b_1, a_2, c_2, d, c_3, a_3)$, leaving $\alpha_1 + \beta_1 \geq 7$. If $\gamma_1 = 1$, there are two cases to consider. If $\alpha_2 + \gamma_2 = 3$, move a pebble from T_2 to d. Then move a pebble along (b_1, c_1, d, c_3, a_3), leaving $\alpha_1 + \beta_1 \geq 8$. If $\alpha_2 + \gamma_2 = 2$, then $\alpha_2 = \gamma_2 = 1$ and $p_1 \geq 12$. Move a pebble along (b_1, a_2, c_2, d) and then (b_1, c_1, d, c_3, a_3), leaving $\alpha_1 + \beta_1 \geq 7$. In all of these cases, Lemma 13.1 allows a second pebble to be moved to a_3.

Case 2.3: $\gamma_3 = \alpha_2 = 0$. Then $\delta = \gamma_2 = \beta_2 = 1$, $\beta_1 \leq 2$, and $\gamma_1 \geq 5$. Move along the path (c_1, d, c_2, b_2, a_3) using only 5 pebbles so Theorem 1 applies.

Case 2.4: $\gamma_3 = \beta_2 = 0$. Then d, a_2, and c_2 each have at least one pebble, and $\delta + \alpha_2 + \gamma_2 \leq 5$, so $p_1 \geq 10$. At most 7 pebbles from T_1 can be used to move a pebble to a_3 through a_1 in such a way that $\beta_1 \geq 1$ and $\gamma_1 \geq 1$ after the moves. Then $\beta_1 + \gamma_1 + d + a_2 + c_2 \geq 8$. Since the graph induced by $\{b_1, c_1, d, a_2, b_2, c_2\}$ is Hamiltonian, Lemma 3 applies. Thus, two pebbles can be moved to b_2, and a second pebble to a_3.

Case 3: $q = 5$. Then $p = 16$. If $\alpha_1 \geq 2$, then 6 pebbles from T_1 can be used to move to a_3. Thus, $\alpha_1 = 1$. There are three cases to consider, each using Corollary 5 extensively.

Case 3.1: $\beta_1 \geq 3$ and $\gamma_1 \geq 3$. Then $1 \leq \alpha_2 \leq 2$ and $1 \leq \delta \leq 2$ and the rest of the vertices (besides a_1) have no pebbles. So $p_1 \geq 12$ and either $\beta_1 \geq 6$ or $\gamma_1 \geq 6$. If $\beta_1 \geq 6$, then $\alpha_2 = 1$ by Theorem 1, and either $\delta = 1$ and $p_1 = 14$ or $\delta = 2$ and $p_1 = 13$ and a move from d to c_1 would make $p = 14$. In either case, Lemma 13.2 applies. Otherwise, $\gamma_1 \geq 6$ and a similar argument applies.

Case 3.2: $\gamma_1 \leq 2$. Then $\beta_1 \geq 5$, $\alpha_2 = 1$, $\beta_2 = 0$, and $\gamma_2 \leq 1$ by Theorem 1. Since $q = 5$, either $\gamma_2 = 1$, $\gamma_3 = 1$, $\delta = 1$, or $\delta = 2$. In the first 3 cases, $p_1 = 14$. In the last case, a move from d to c_1 makes $p_1 = 14$. In all cases, $\beta_1 \geq 10$, so Lemma 13.2 applies.

Case 3.3: $\beta_1 \leq 2$. Then $\gamma_1 \geq 5$, so $\delta = 1$, $\gamma_3 = 0$, and $\gamma_2 \leq 1$ by Theorem 1. Since $q = 5$, either $\beta_2 = 1$, $\gamma_2 = 1$, $\alpha_2 = 1$, or $\alpha_2 = 2$. In the first three cases, $p_1 = 14$. If $\alpha_2 = 2$, a move from a_2 to b_1 leaves $p_1 = 14$. In all cases, $\gamma_1 \geq 10$, so Lemma 13.2 applies.

Case 4: $q = 4$. Then $p = 17$ and Theorem 1 and Corollary 5 imply that either $\beta_1 = 14$ and $\gamma_1 = \alpha_1 = \alpha_2 = 1$; $\beta_1 = 13$, $\gamma_1 = 2$, and $\alpha_1 = \alpha_2 = 1$; $\gamma_1 = 14$ and $\alpha_1 = \beta_1 = \delta = 1$; or $\gamma_1 = 13$, $\beta_1 = 2$, and $\alpha_1 = \delta = 1$. In all cases, Lemma 13.2 applies.

Lemma 16. *Let C be a configuration on H. Then for $i \in \{1, 2, 3\}$, the following hold.*

1. *If $p_i \geq 8$, or $p_i = 7$ and $q_i = 2$, then a pebble can be moved to c_3.*
2. *If $\delta = 1$ and $p_i + q_i \geq 13$, two pebbles can be moved to c_3.*
3. *If $p_i + q_i \geq 17$, two pebbles can be moved to c_3.*

Proof. The statements are obvious when $i = 3$. Statement 1 follows from Lemma 6.5. To prove statement 2, when $q_i = 1$, use at most 4 pebbles from T_i to move to c_3, leaving 8 on some vertex of distance at most 3 from c_3. For $q_i = 2$ and $q_i = 3$, make moves from T_1 to d to c_3, and apply statement 1. For statement 3, apply statement 1 for $q_i = 1,2$ and use Lemma 6.8 for $q_i = 3$.

Theorem 17. *H has exactly two violating configurations with c_3 as the root.*

Proof. Let C be a violating configuration with $21-q$ pebbles on q vertices. By Lemma 11, we only need to consider $4 \le q \le 7$. In all of these cases, $p \ge 14$. By Theorem 1, c_3 has no pebbles, a_3, b_3, and d each have at most one pebble, and a_1, c_1, c_2, and b_2 each have at most three pebbles. We can assume that $p_1 \ge p_2$. Since $p_1 + p_2 \ge 11$, $p_1 \ge 6$. If both a_1 and b_3 have at least one pebble, a pebble can be moved to c_3 using four pebbles. Thus, at least one of a_1 or b_3 has no pebbles. Similarly, at least one of c_1 or d has no pebbles.

Case 1: $q = 7$. Then $p = 14$, and since two of a_1, b_3, c_1, and d have no pebbles, each of b_1, a_2, b_2, c_2, and a_3 has at least one pebble. Theorem 1 implies that $\alpha_2 = \beta_2 = \gamma_2 = \alpha_3 = 1$. Clearly $\alpha_1 + \beta_3 \le 3$ and $\gamma_1 + \delta \le 3$, so $\beta_1 \ge 4$. Pebble along the path $(b_1, a_2, b_2, a_3, c_3)$, leaving 8 pebbles on $\{a_1, b_1, c_1, b_3, d\}$. Since the subgraph induced by $\{a_1, b_1, c_1, b_3, c_3, d\}$ has C_6 as a spanning subgraph, and $\pi(C_6) = 8$, a second pebble can be moved to c_3.

For the remainder of the cases, $q \le 6$ and $p \ge 15$. Since $p_1 \ge 6$, $\delta = \beta_3 = 0$ by Theorem 1. This and the fact that $\alpha_3 \le 1$ implies that $p_1 + p_2 \ge 14$, so $p_1 \ge 7$. Corollary 5 implies that $1 \le \alpha_1 \le 2$ and $1 \le \gamma_1 \le 2$, so $\beta_1 \ge 3$, $q_1 = 3$, and $\alpha_2 \ge 1$. By Theorem 1, at least one of b_2 or a_3 has no pebbles.

Case 2: $q = 6$. Then $p = 15$, exactly one of b_2 or a_3 has no pebbles, and $\gamma_2 = 1$ by Theorem 1.

Case 2.1: $\beta_2 = 0$. Then $1 \le \alpha_2 \le 2$ by Corollary 5 and $p_1 \ge 11$. If $\alpha_2 = 2$, move a pebble along the path (a_2, c_2, d). If $\alpha_2 = 1$, then $p_1 = 12$ and we move a pebble along the path (b_1, a_2, c_2, d). In both cases, $p_1 \ge 10$ and $\delta = 1$ after the moves, so Lemma 16.2 applies.

Case 2.2: $\alpha_3 = 0$. Since $q_2 = 3$ and $p_1 \ge p_2$, $3 \le p_2 \le 7$. Since $\alpha_3 = \beta_3 = \delta = 0$, Corollary 5 implies that each of α_1, γ_1, and β_2 is 1 or 2.

Case 2.2.1: $3 \le p_2 \le 6$. If $p_2 = 3$, then $a_2 = 1$, and $p_1 = 12$. Move a pebble along the path (b_1, a_2, c_2, d). If $4 \le p_2 \le 5$, then $p_1 \ge 10$ and since $q_2 = 3$, a pebble can be moved from T_2 to d. If $p_2 = 6$, $p_1 = 9$, and since $q_2 = 3$, a pebble can be moved to both d and b_1 from T_2. In all three cases, $p_1 \ge 10$, $q_1 = 3$, and $\delta = 1$ after the moves, so Lemma 16.2 applies.

Case 2.2.2: $p_2 = 7$. Then $p_1 = 8$ and c_3 can be reached from T_1 by Lemma 16.1. If $\beta_2 = 2$, c_3 can also be reached from T_2. Thus $\gamma_2 = \beta_2 = 1$ and $\alpha_2 = 5$. If $\gamma_1 = 2$, then c_3 can be reached using 5 pebbles from c_1, a_2, and c_2, so Theorem 1 applies. Thus $\gamma_1 = 1$. If $\alpha_1 = 2$, then $\beta_1 = 5$. Move two pebbles from b_1 to a_1 and then through b_3 to c_3. Then move along the paths (a_2, b_1, c_1, d), (a_2, c_2, d), and from d to c_3 with a second pebble. This implies that $\alpha_1 = \gamma_1 = \gamma_2 = \beta_2 = 1$, $\beta_1 = 6$, and $\alpha_2 = 5$. Because of the symmetry of the graph, if we remove our assumption that $p_1 \ge p_2$, $\beta_1 = 5$ and $\alpha_2 = 6$ also leads to a violating configuration. It is easy to see that if we add a pebble to either b_1 or a_2, it is possible to move two pebbles to c_3. Thus, when $q = 6$ there are exactly two violating configurations with $p = 15$, and none with $p \ge 16$.

Case 3: $q = 5$. Then $p = 16$ and $p_1 \ge 8$, so Theorem 1 implies $\gamma_3 = \delta = \beta_3 = 0$, $\alpha_1 = \gamma_1 = 1$, and $\gamma_2 \le 1$. Exactly one of a_3, b_2, and c_2 has any pebbles.

Notice that there are exactly 16 configurations of pebbles with $q = 5$ and $p = 16$ that yield one of the violating configurations above after a move is made (8 for each), and it is easy to check that they are not violating configurations before the move. For instance, if $\beta_1 = 8$, $\alpha_2 = 5$, and $\alpha_1 = \gamma_2 = \beta_2 = 1$, move from b_1 to a_2 and apply Lemma 16.1 to both T_1 and T_2. Similarly for $\beta_1 = 6$, $\alpha_2 = 5$, $\alpha_1 = 3$, and $\gamma_2 = \beta_2 = 1$. Lemma 4 implies that we can assume for the remainder of the cases that if $C(v) = 0$, then $C(u) \le 2$ for any neighbor u of v. Thus, $1 \le \alpha_2 \le 2$ since at least one of its neighbors has no pebbles. Similarly, $\beta_2 \le 2$.

If $\beta_2 = 2$, then $\gamma_2 = \alpha_3 = 0$ and $\beta_1 \ge 10$. If $\alpha_2 = 2$, move from both a_2 and b_2 to c_2 and then to d. If $\alpha_2 = 1$, then $\beta_1 = 11$. Pebble along the path (b_1, a_2, c_2) and then (b_2, c_2, d) leaving $\beta_1 = 9$. In either case, Lemma 16.2 applies.

If $\beta_2 \le 1$, then $\beta_2 + \gamma_2 + \alpha_3 = 1$. Thus, either $\beta_1 = 12$ and $\alpha_2 = 1$, or $\beta_1 = 11$ and $\alpha_2 = 2$ and we move one pebble from a_2 to b_1. In both cases, Lemma 16.3 applies.

Case 4: $q = 4$. Then $p = 17$. Corollary 5 and Theorem 1 imply that $\alpha_1 = \gamma_1 = 1$, $1 \le \alpha_2 \le 2$, and $13 \le \beta_1 \le 14$, and Lemma 16.3 applies.

Given the symmetry of the graph, the following result is obvious.

Theorem 18. *H has exactly* 6 *violating configurations.*

Theorem 19. *H does not have the two-pebbling property, but does have the odd-two-pebbling property.*

Proof. H does not have the two-pebbling property by Theorem 18. All of the violating configurations have $p = 15$ and $r = 5$, and since $15 \not\ge 20 - 5 = 2\pi(H) - r$, they do not violate the odd-two-pebbling property.

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