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Doppelgangers and Lemke graphs

Charles A. Cusack [a,b,*], Mark Powers [a], Airat Bekmetjev [b]

[a] Department of Computer Science, Hope College, 27 Graves Place, Holland, MI 49422, United States
[b] Department of Mathematics, Hope College, 27 Graves Place, Holland, MI 49422, United States

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ABSTRACT

Let G be a connected graph. A configuration of pebbles on G is a function that assigns a nonnegative integer to each vertex. A pebbling move consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. A configuration is solvable if after making pebbling moves any vertex can get at least one pebble. The pebbling number of G, denoted $\pi(G)$, is the smallest integer such that any configuration of $\pi(G)$ pebbles on G is solvable. A graph has the two-pebbling property if after placing more than $2\pi(G) - q$ pebbles on G, where q is the number of vertices with pebbles, there is a sequence of pebbling moves so that at least two pebbles can be placed on any vertex. A graph without the two-pebbling property is called a Lemke graph. Previously, an infinite family of Lemke graphs was shown to exist by subdividing edges of the original Lemke graph. In this paper, we introduce a new way to create infinite families of Lemke graphs based on adding vertices as well as subdividing edges. We also characterize the configurations that violate the two-pebbling property on these graphs and conjecture another infinite family of Lemke graphs that generalizes the original Lemke graph.

1. Introduction

In this paper we assume all graphs are connected. A *configuration* C on G is a function from $V(G)$ to the nonnegative integers. $C(v)$ represents the number of pebbles on a vertex. A pebbling move from vertex u to an adjacent vertex v consists of removing two pebbles from u and adding one pebble to v. A configuration is *r-solvable* if there is a sequence of pebbling moves so that at least one pebble can reach the target vertex r. A configuration is *solvable* if it is r-solvable for all vertices. The *pebbling number rooted at a vertex r* in G, $\pi(G, r)$, is defined as the fewest number of pebbles so that for any configuration of $\pi(G, r)$ pebbles the graph is r-solvable. The pebbling number of a graph is $\pi(G) = \max_{r\in V(G)}(\pi(G, r))$. It is evident that $\pi(G) \geq 2^{diam(G)}$.

G has the *two-pebbling property* if any configuration of more than $2\pi(G) - q$ pebbles on G makes it possible to get two pebbles to any target vertex, where q is the number of vertices with pebbles. A *violating configuration* on a graph G is a configuration of more than $2\pi(G) - q$ pebbles, where q is the number of vertices with pebbles, such that it is impossible to move two pebbles to some vertex in G.

A graph is a *Lemke graph* if it does not have the two-pebbling property. In other words, it is a graph that has at least one violating configuration. Originally there was one known Lemke graph L [2]. To help make the structure of L and other Lemke graphs more evident, we have chosen to draw them in a different arrangement than has been usually done (see Fig. 1).

First conjectured by Snevily and Foster [11], L_t (Fig. 2), that replaces each of the edges (a, x), (b, x), (c, x), and (d, x) in L with a path of length t for $t \geq 0$ was shown to be a Lemke graph [5]. Additionally, it was shown that a modification of this, L'_t (Fig. 3), where the corresponding vertices from the added paths in L_t are adjacent, is a Lemke graph [12].

* Corresponding author at: Department of Computer Science, Hope College, 27 Graves Place, Holland, MI 49422, United States.
E-mail addresses: cusack@hope.edu (C.A. Cusack), bekmetjev@hope.edu (A. Bekmetjev).

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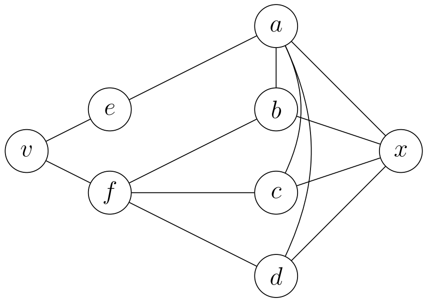

Fig. 1. The original Lemke graph L.

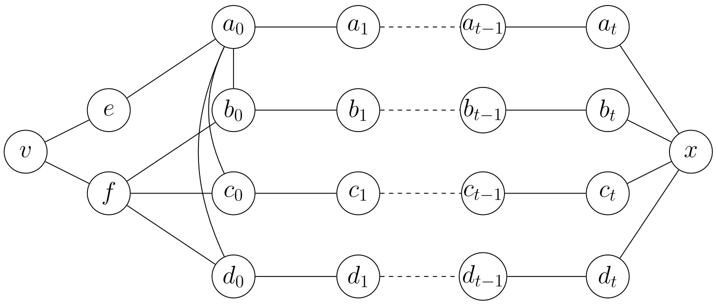

Fig. 2. The graph L_t.

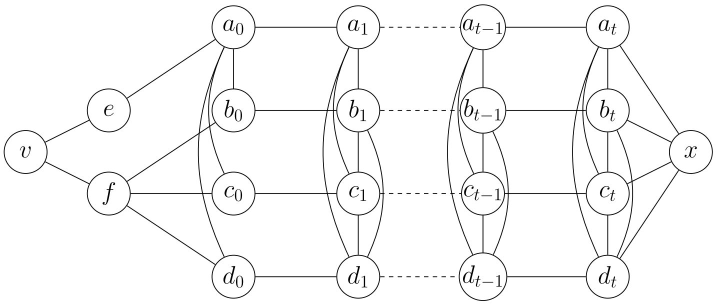

Fig. 3. The graph L'_t.

For graphs G and H, let $H \subseteq G$ denote that H is a subgraph of G. Gao and Yin defined the set $A(t) = \{G \mid L_t \subseteq G \subseteq L'_t\}$ for $t \geq 0$ [6]. Until recently, the set of all previously known Lemke graphs was $\cup_{t=0}^{\infty} A(t)$. Note that for any graph $G \in A(t)$, $\pi(G) = 2^{t+3}$. In particular, $\pi(L) = 8$.

A longstanding conjecture in graph pebbling is Graham's Conjecture [2]. It states for any two graphs G and H, $\pi(G \square H) \leq \pi(G)\pi(H)$, where $G \square H$ is the Cartesian product of G and H. In many results that support Graham's Conjecture, the two-pebbling property of a graph is used. If Graham's Conjecture is false, Lemke graphs are suspected to be involved in a counterexample. In particular, $L \square L$ has been of interest [8].

In [4], all non-isomorphic graphs with at most 9 vertices were tested for the two-pebbling property. It was discovered that there are no Lemke graphs with fewer than 8 vertices, 22 Lemke graphs with 8 vertices, and 306 Lemke graphs with 9 vertices. Note that only some of the 8-vertex Lemke graphs and none of the 9-vertex Lemke graphs are in $\cup_{t=0}^{\infty} A(t)$. After analyzing these graphs, we have found a new way to construct Lemke graphs. Given any diameter $d \geq 3$, a Lemke graph can be constructed with at least $4(d-1)$ vertices, producing many previously unknown Lemke graphs.

2. Doppelgangers

To help construct a new family of Lemke graphs, we define a way to add vertices to a graph. Two vertices are *doppelgangers* if they have the same adjacency list. Let G be a graph with vertex v. Define a *doppelganger graph*, $D(G, v, k)$, to be the graph G

with k additional vertices (called *doppels*) that are doppelgangers of vertex v (called the *ganger*). The *complete doppelganger graph*, $D'(G, v, k)$, is constructed starting with $D(G, v, k)$ and adding edges between all of the doppels and the ganger to form a clique.

Theorem 1. *Let G be a graph with at least 3 vertices and let $v \in V(G)$. Then for $k \geq 1$, $\pi(G) \leq \pi(D'(G, v, k)) = \pi(D(G, v, k)) \leq \pi(G) + k$.*

Proof. Clearly, any configuration that is unsolvable on G is also unsolvable on $D(G, v, k)$. Therefore, $\pi(G) \leq \pi(D(G, v, k))$.

Let C be a configuration with $\pi(G) + 1$ pebbles on $D(G, v, 1)$ and without loss of generality assume $C(d) \leq C(v)$, where d is the doppel of v. Note that if both v and d are reachable under C, then an even number of pebbles can be removed from one and placed on the other without changing the solvability of C on the rest of the graph. Thus, if $C(d) = 2l$ or $C(d) = 2l+1$ for $l \geq 1$, remove $2l$ pebbles from d and place $2l$ pebbles on v. Then there are at least $\pi(G)$ pebbles on the subgraph G, so the configuration is solvable which implies C is solvable. If $C(d) = 1$, then there are $\pi(G)$ pebbles on the subgraph G, so the configuration is solvable.

Assume $C(d) = 0$. We then have $\pi(G) + 1$ pebbles on G and we only need to show that d is reachable. If $C(v) \geq 4$, then d is clearly reachable from v.

If $C(v) = 3$ and any of the neighbors of v has at least 1 pebble (or at least one pebble can be moved to it without using any pebbles from v) then another pebble can be moved to the same vertex from v making d reachable.

Let $N(v)$ be the set of all vertices adjacent to v and assume that $C(u) = 0$ for all $u \in N(v)$ and none of the vertices in $N(v)$ are reachable except from v. Consider a pebbling configuration C' where a pebble is moved from v to a vertex $x \in N(v)$. There are $\pi(G)$ pebbles on the subgraph G in C', so all neighbors of v are reachable in C'. If $|N(v)| \geq 2$, then in order to reach any vertex $u \neq x$ in $N(v)$ any sequence of pebbling moves in C' has to move through x. This implies that at least 2 pebbles can be moved to x making d reachable in C' and in C.

If v has only one neighbor x, then since $|V(G)| \geq 3$ we know x has at least one neighbor y aside from v and d. Consider a new configuration C'' on G by removing all pebbles from v and placing 2 more pebbles on y. Then in C'' at most 3 pebbles can be moved to y and at most 1 pebble on each other neighbor of x, which allows us to move at most 1 pebble to x, making v unreachable. However, this is a contradiction, since there are $\pi(G)$ pebbles on the vertices of the subgraph G. Therefore, we can get at least 1 pebble to x in C and 1 more from v making vertex d reachable.

If $C(v) = 2$, move a pebble from v to a neighbor. Now there are $\pi(G)$ pebbles on the subgraph G with none on v, so v is reachable, therefore d is as well.

If $C(v) \leq 1$, then there are $\pi(G)$ pebbles elsewhere on the subgraph G, which means v is reachable with these pebbles, and therefore d is too.

Thus, $\pi(D(G, v, 1)) \leq \pi(G) + 1$. Since for $k \geq 2$, $D(G, v, k) = D(D(G, v, k-1), v, 1)$, $\pi(D(G, v, k)) \leq \pi(G) + k$.

Next we prove that $\pi(D'(G, v, k)) = \pi(D(G, v, k))$ for $k \geq 1$. Clearly $\pi(D'(G, v, k)) \leq \pi(D(G, v, k))$. Suppose $\pi(D'(G, v, k)) < \pi(D(G, v, k))$. Let D be the set of k doppels and let C be a configuration with $\pi(D'(G, v, k))$ pebbles that is unsolvable on $D(G, v, k)$. This implies that on $D'(G, v, k)$, without loss of generality, a move along the edge (d, v) is necessary to reach some vertex r, for some $d \in D$. If $r \neq v$, then a move along the edge (v, x) is required for some neighbor x of v. But since d is a doppelganger of v, the moves from d to v and v to x can be replaced with a single move from d to x, making the move along (d, v) unnecessary. Therefore, $r = v$. This implies that $C(v) = 0$. Further, $2 \leq C(d) \leq 3$ and $C(d') \leq 1$ for any $d' \in D \setminus \{d\}$ since otherwise v can be reached without moving along the edge (d, v).

If v has one neighbor x in $D(G, v, k)$, then $C(x) = 0$ and at most one pebble can be moved to each neighbor of x not in $D \cup \{v\}$. Let C' be the configuration C modified by removing 2 pebbles from d and adding 2 pebbles to a neighbor $u \notin D \cup \{v\}$ of x. Then u can receive at most 3 pebbles and all other neighbors of x can still receive at most one pebble under C'. Therefore, even though C' has $\pi(D'(G, v, k))$ pebbles on the vertices of $D'(G, v, k)$, v is not reachable under C', a contradiction.

If v has more than one neighbor in $D(G, v, k)$, each of these must have 0 pebbles in C and they can only receive a pebble from d. Let C' be the configuration C modified by removing 2 pebbles from d and placing one on each of any two neighbors of v not in D. Then no additional pebble can be placed on any neighbor of v, so v is not reachable on $D'(G, v, k)$ under C'. But again, C' has $\pi(D'(G, v, k))$ pebbles on the vertices of $D'(G, v, k)$, a contradiction. Therefore, C cannot be unsolvable on $D(G, v, k)$, so $\pi(D(G, v, k)) = \pi(D'(G, v, k))$. □

3. Lemke graphs on 8 vertices

There are 22 Lemke graphs on 8 vertices, all of which are subgraphs of A (Fig. 4). An interesting property of A is that it has subgraphs K_3 and K_5. More will be said about this later. Since the original Lemke graph is a spanning subgraph of A, $\pi(A) = 8$.

There are three minimal Lemke graphs on 8 vertices that have no subgraphs in the set of 22. One of these is the original Lemke graph, L (Fig. 1). Additionally, there is B, which is the minimum Lemke graph with respect to number of edges and vertices. The final graph, C, exhibits symmetry, with the vertices v and x being equivalent to their counterparts e and c, respectively. These graphs are shown in Fig. 5.

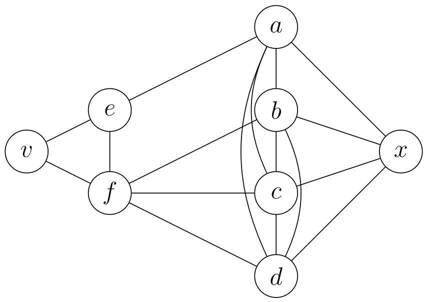

Fig. 4. The maximal 8-vertex Lemke graph A.

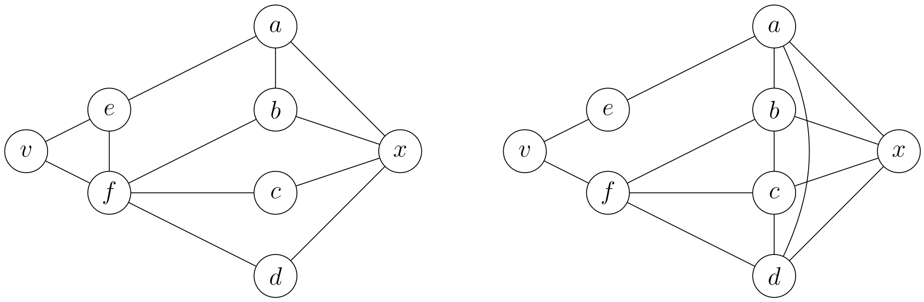

Fig. 5. The minimal Lemke graph B and a symmetric Lemke graph C.

Using Algograph [3], it was verified computationally that $\pi(B) = \pi(C) = 8$. The proofs that $\pi(B) = 8$ and $\pi(C) = 8$ can be approached by exhausting cases in several different ways or by arguing based on the difference between these graphs and L. Both approaches are straightforward, but tedious and not very enlightening, so they are omitted.

Consider the configuration on A of one pebble on a, b, c, and d and 9 pebbles on x. It is easy to see that this configuration does not allow moving two pebbles to v, so A is a Lemke graph.

The following lemma is straightforward.

Lemma 2. *Let G be a Lemke graph. Let H be any spanning subgraph of G with $\pi(H) = \pi(G)$. Then H is a Lemke graph.*

Since B and C are spanning subgraphs of A with the same pebbling number, Lemma 2 implies that they are Lemke graphs.

The remaining 18 Lemke graphs on 8 vertices share similar structure. The only difference in these graphs comes about in variations of edges between vertices a, b, c, and d along with the presence or absence of the edge (e, f). At a minimum, a, b, c, and d are connected with a star centered at a or a cycle, or both of the edges (e, f) and (a, b) are present. One of these structures is required so that the pebbling number of the graph does not exceed 8.

All 22 of these graphs have the same six violating configurations. In addition to the configuration with one pebble on a, b, c, and d and 9 pebbles on x, the number of pebbles on x can be reduced to 8 and it is still a violating configuration. The other four configurations are the result of making a move from x to one of its neighbors on the first configuration. This leads to, for instance, a configuration with 1 pebble on a, b, and c, 2 on d, and 7 on x. Two of the graphs, including C, have an additional six violating configurations because they exhibit symmetry. They are basically the same set of configurations but with a relabeling of four of the vertices as alluded to earlier. Note that all violating configurations have pebbles on 5 vertices.

4. Lemke graph subdivision

Let L_t'' be the graph A subdivided along the edges (a, x), (b, x), (c, x), and (d, x) with a path of length t where corresponding vertices from each path are adjacent as shown in Fig. 6. Note that this is equivalent to L_t' plus the edges (e, f), (b_0, c_0), (b_0, d_0), and (c_0, d_0).

Theorem 3. *L_t'' is a Lemke graph for all $t \geq 0$.*

Proof. Since L_t is a spanning subgraph of L_t'', $\pi(L_t'') \leq \pi(L_t) = 2^{t+3}$. The configuration with $2^{t+3} - 1$ pebbles on x is unsolvable for v, thus, $\pi(L_t'') = 2^{t+3}$. Place the configuration of one pebble on a_t, b_t, c_t, d_t and $2^{t+4} - 7$ pebbles on x on L_t''. It is clear that this configuration does not allow two pebbles to be placed on v. □

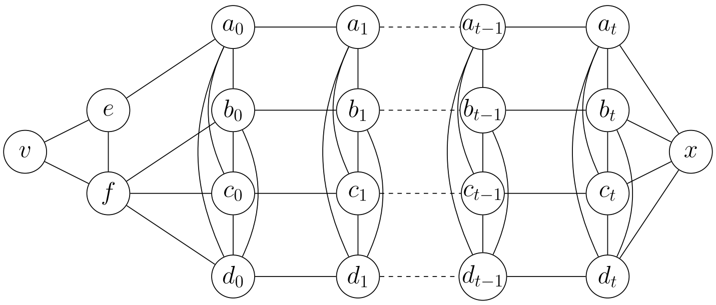

Fig. 6. L''_t.

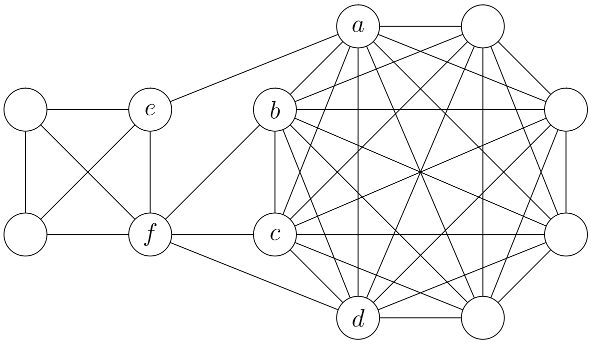

Fig. 7. $L_{8,4}$, a Lemke graph constructed from K_4 and K_8.

From this result, the set of Lemke graphs can easily be expanded to include $S(t) = \{G \mid L_t \subseteq G \subseteq L''_t\}$ for $t \geq 0$, but we will expand it even further. Consider $G \in S(t)$ for some $t \geq 0$. Note, $\pi(G) = 2^{t+3}$. Define $G(i, j)$ to be $D'(D'(G, x, i), v, j)$.

Theorem 4. *Let $G \in S(t)$ for some $t \geq 0$. Then $\pi(G(i, j)) = 2^{t+3} + i + j$ and $G(i, j)$ is a Lemke graph for all $i \geq 0, j \geq 0$.*

Proof. By Theorem 1, $\pi(G(i, j)) \leq 2^{t+3} + i + j$. Consider the configuration with 1 pebble on all doppels of x, one pebble on all doppels of v, and $2^{t+3} - 1$ pebbles on x. With this configuration, it is clear that v is not reachable, so $\pi(G(i, j)) = 2^{t+3} + i + j$.

Place on $G(i, j)$ the configuration of 1 pebble on a_t, b_t, c_t, d_t, all doppels of x, and all doppels of v. Additionally, place $2^{t+4} - 7$ pebbles on x. Then we have $2^{t+4} + i + j - 3 > 2\pi(G(i, j)) - (i + j + 5)$ pebbles on $(i + j + 5)$ vertices. The doppels of x and v, along with their pebbles, can be ignored since any sequence of moves through them can be replaced with a shorter sequence. For instance, instead of pebbling from x to a doppel and then to a_t, a move can be made directly from x to a_t. Ignoring these pebbles and vertices, we are left with the same configuration on G that does not allow pebbling twice to v, thus it is not possible on $G(i, j)$. □

Theorem 4 leads to a very simple method of creating Lemke graphs by connecting two cliques of sizes at least 3 and 5 with 4 edges. This is done as follows. Let K_i and K_j be complete graphs with $i \geq 5$, and $j \geq 3$. Let $a, b, c, d \in V(K_i)$ and $e, f \in V(K_j)$. Construct $L_{i,j}$ by adding edges (e, a), (f, b), (f, c), and (f, d). An example is shown in Fig. 7.

Corollary 5. *Let $G \in S(t)$ for some $t \geq 0$. Let H be a graph such that $D(D(G, x, i), v, j) \subseteq H \subseteq D'(D'(G, x, i), v, j)$ for some $i \geq 0$, $j \geq 0$. Then H is a Lemke graph.*

Proof. By Theorem 1, $\pi(H) = 2^{t+3} + i + j$. Since H is a spanning subgraph of $D'(D'(G, x, i), v, j)$, by Lemma 2, H is a Lemke graph. □

Theorem 6. *Let G be a graph (connected or not). Then there exists a Lemke graph with G as an induced subgraph.*

Proof. Let $|V(G)| = n$ and construct $D(A, x, n - 1)$. Place a copy of G on the $n - 1$ doppels and x. By Corollary 5, this is a Lemke graph. □

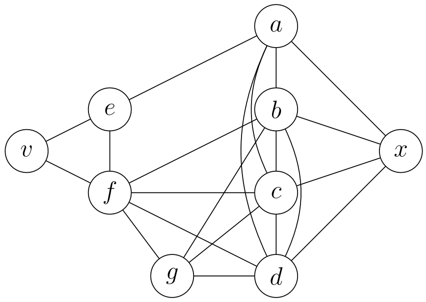

Fig. 8. A maximal Lemke graph with 9 vertices.

5. Lemke graphs on 9 vertices

Of the 306 9-vertex Lemke graphs, 212 have induced subgraphs of Lemke graphs on 8 vertices. Among these, 88 are 8-vertex Lemke graphs plus a doppelganger of x or v. There are 3 maximal Lemke graphs on 9 vertices such that all other Lemke graphs on 8 and 9 vertices are subgraphs of them. Two of these are $L_{5,4}$ and $L_{6,3}$. The third is A with an additional vertex that is connected to b, c, d and f (Fig. 8). It is tedious but straightforward to verify that the pebbling number of this graph is 9. When one pebble is placed on a, b, c, d, and g, and 9 pebbles on x, it is impossible to move two pebbles to v, thus it is a Lemke graph.

The minimal structure for Lemke graphs on 9 vertices allows for many more possibilities than the minimal Lemke graphs of size 8. There is a total of 72 Lemke graphs on 9 vertices that do not have any of the 8-vertex Lemke graphs as a subgraph. We consider twelve of them to be minimal since they do not have other 8- or 9-vertex Lemke graphs as subgraphs. Another 7 of them have an 8-vertex Lemke graph as an induced subgraph, but no other 9-vertex Lemke graphs as a subgraph. It is interesting to note that all violating configurations on graphs with 9 vertices place pebbles on 6 vertices.

6. Violating configurations

Because many results related to Graham's conjecture involve products of graphs that have the two-pebbling property, proving the conjecture for products involving Lemke graphs can often be boiled down to dealing with just the violating configurations. For instance, Gao and Yin proved that Graham's conjecture holds for the product of Lemke graphs with both trees and complete graphs by addressing the violating configurations [6]. Thus, it can be important to be able to identify all violating configurations of a Lemke graph.

Let $G \in S(t)$ for some $t \geq 0$. Here we consider violating configurations on the graph $G(i, j)$ for $i \geq 0$ and $j \geq 0$. From Theorem 4, $\pi(G(i, j)) = 2^{t+3} + i + j$. Let $X = \{x_1, x_2, \ldots, x_{i+1}\}$ denote the set of x and its doppels (where $x = x_1$) and let V denote the set of v and its doppels. Let $r \in V$. Construct a configuration on $G(i, j)$ as follows. Place 1 pebble on each vertex a, b, c, and d and each vertex in $V \setminus \{r\}$. For any odd partition (i.e. each p_k is odd) $p_1 + \cdots + p_{i+1} = i + 2^{t+4} - 7$, place p_k pebbles on vertex $x_k \in X$. Then there are $4 + j + (i + 2^{t+4} - 7) = 2\pi(G(i, j)) - (i + j + 5) + 2$ pebbles on $i + j + 5$ vertices.

From the pebbles on vertices in X, at most $((i + 2^{t+4} - 7) - (i + 1))/2 = 2^{t+3} - 4$ pebbles can be moved to vertices a, b, c, and d. Notice that this the same number of pebbles that can be moved to vertices a, b, c, and d from the configuration in Theorem 4 and that the pebbles on $V \setminus \{r\}$ are of no use in getting a pebble to r. Thus it is impossible to place two pebbles on r so the configuration described is a violating configuration for $G(i, j)$.

Since the number of pebbles on this configuration is one greater than the minimum number required for a violating configuration, we can remove one pebble from a single vertex in X that has at least two pebbles or make a move from a vertex in X to a vertex in $\{a, b, c, d\}$. Either of these results in one fewer pebble in the configuration without changing q, thus it still violates the two-pebbling property. Notice that the 6 violating configurations on L are included in this description.

Conjecture 7. *The configurations described above are all of the configurations that violate the two-pebbling property on $G(i, j)$.*

So far, for every Lemke graph for which all violating configurations are known, they all have the same q value. If the previous conjecture is true, it would not change this observation. However, we recently discovered a Lemke graph on 10 vertices that has violating configurations for both $q = 6$ and $q = 7$ (Fig. 9). The pebbling number of this graph is 11. The first violating configuration places 13 pebbles on x, one on each of b, c, d, g, and h, and zero on the rest of the vertices. The second violating configuration can be obtained by moving a pebble from x to a so that x has 11 pebbles, a, b, c, d, g, and h each have one, and the rest have zero.

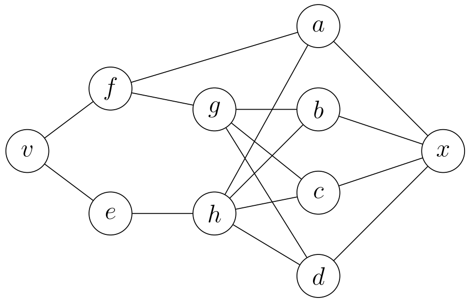

Fig. 9. A Lemke graph that has violating configurations with $q = 6$ and $q = 7$.

7. Additional Lemke graphs

All currently known Lemke graphs seem to rely on the same principle—a partition of the number 4 as 1+3. The key is that enough pebbles can be placed next to the root so that two pebbles should be able to be placed on the root, but because three are placed on one vertex and one on the other vertex, this turns out to not be possible. This may lead to a generalization.

Let $k \geq 2$, $p_1 + \cdots + p_l = 2^k$ be a partition of 2^k, and let $P = \{p_1 \ldots p_l\}$. Construct graph $J_{k,P}$ as follows. Begin with vertices x and $Z = \{z_1, \ldots, z_l\}$, a path of length $k - 2$ with vertices $v_0, v_1, \ldots, v_{k-2}$, and a complete graph K_{2^k}. Connect x to every vertex in K_{2^k} and connect v_0 to every vertex in Z. Finally, for $1 \leq i \leq l$ connect z_i to p_i vertices from K_{2^k} such that each vertex K_{2^k} is connected to a single vertex in Z. Notice that $J_{2,\{1,3\}}$ is the graph A minus the edge (e, f) and is a Lemke graph.

Conjecture 8. *If P has at most 2^{k-1} parts, $\pi(J_{k,P}) = 2^{k+1}$. If P has $2^{k-1} + b$ parts, where $b \geq 1$, then $\pi(J_{k,P}) = 2^{k+1} + b$.*

Next we argue that if Conjecture 8 is true, then certain $J_{k,P}$ are Lemke graphs by demonstrating a violating configuration. Consider $J_{k,P}$ where P contains at least one odd p_i with a configuration that has one pebble on each vertex of K_{2^k} and 2^{k+1} pebbles on x. We will show that two pebbles cannot be moved to v_{k-2} by using a weight function argument.

Given a root vertex r, define $w(u) = 2^{-dist(u,r)}$ for each vertex u and for a given configuration C, define $w(C) = \sum_{u \in V(G)} C(u)w(u)$. Notice that a pebbling move can never increase the weight of a configuration. Thus, if $w(C) < 1$, then r is unreachable [1,7,9]. Also, if $w(C) = 1$, all pebbles will be used to reach r and all moves made must be toward r.

With the above configuration C and root v_{k-2}, $w(C) = 2$. Thus, reaching v_{k-2} with two pebbles requires every pebble on the graph and every move made must go toward the root. Thus the pebbles from x must be distributed to each vertex of K_{2^k} and then a move from each of those vertices to the vertices in Z is necessary. Since at least one z_i has an odd number of pebbles, we are able to get at most $2^{k-1} - 1$ pebbles to v_0 which is distance $k - 2$ from v_{k-2}. Thus, it is impossible to move 2 pebbles to v_{k-2}.

If P contains at least three odd p_i, consider the configuration with one pebble on each vertex of K_{2^k} and $2^{k+1} + 2$ pebbles on x. For this configuration C and root v_{k-2}, $w(C) = 2 + 1/2^k$. This much additional weight allows us to either leave two pebbles on x, one pebble on K_{2^k}, or to make at most one move that is not toward the root. It is easy to see that the first and third options are not helpful in reaching v_{k-2}. It should also be easy to see that making the obvious moves to use all pebbles on K_{2^k} does not work, even with the additional two pebbles on x. Assume p_1, p_2, and p_3 are odd. Then the only other option is to skip moving one pebble to a vertex in K_{2^k} that is connected to z_1 and instead move a second and third pebble to a vertex in K_{2^k} that is connected to z_2. This allows us to place 2^k pebbles on vertices that are of distance $k - 1$ from v_{k-2}, with an even number on z_1 and z_2, but there is still an odd number on v_3, so reaching v_{k-2} with two pebbles is still impossible as argued above.

The other $J_{k,P}$ do not seem to be Lemke graphs. When P has 2^{k-1} parts and none of p_i are odd, or when P has $2^{k-1} + 1$ parts and at most two of the p_i are odd, the configurations described above do not prevent two pebbles from reaching v_{k-2}. When P has more than $2^{k-1} + 1$ parts, the increase in pebbling number (assuming Conjecture 8 is true) seems to prevent the creation of any violating configurations. This leads to the following conjecture.

Conjecture 9. *$J_{k,P}$ is a Lemke graph if and only if P has at most 2^{k-1} parts and at least one p_i is odd or P has $2^{k-1} + 1$ parts and at least three p_i are odd.*

We should note that for all vertices except $v_0, \ldots, v_{k-2}$, it is not too difficult to determine that Conjecture 8 provides an upper bound on the pebbling number. Thus, proving Conjecture 9 boils down to determining the pebbling number of the vertices $v_0, \ldots, v_{k-2}$.

It is not difficult to see that the each of the configurations described above are still violators if one additional pebble is placed on x. We can perform the analogous operations on these configurations that we described in the previous section to obtain another 2^k violating configurations. Thus, we expect that in general, $J_{k,P}$ will have at least $2^k + 2$ violating configurations.

We verified computationally (using Algograph [3]) that $J_{3,\{3,5\}}$ and $J_{3,\{1,7\}}$, $J_{3,\{1,1,3,3\}}$, $J_{3,\{1,2,2,3\}}$, $J_{3,\{1,1,1,2,3\}}$, and $J_{3,\{1,1,1,1,4\}}$ are Lemke graphs by computing their pebbling numbers (the first four being 16 and the final two 17). We also used a version of our two-pebbling property algorithm to verify that the configurations described above are indeed violators.

Define $J'_{k,P}$ to be the graph $J_{k,P}$ with edges added between all of the vertices in Z. If $J_{k,P}$ is a Lemke graph, then it is likely that $J'_{k,P}$ is as well since the added edges are of no use in reaching v_{k-2} and they probably do not reduce the pebbling number (the assumption being that v_{k-2} is the hardest to reach). If so, it would certainly seem to be maximal. This would imply that if H is a graph such that $J_{k,P} \subseteq H \subseteq J'_{k,P}$, then H is a Lemke graph. It is likely that many of the edges from K_{2^k} could be removed. Thus, as is the case with $J_{2,\{1,3\}}$, there would likely be multiple minimal Lemke graphs that are subgraphs of $J'_{k,P}$. A full characterization of these graphs will require more work.

We suspect that we will be able to use both doppelgangers and subdivisions on these proposed Lemke graphs to construct a more general infinite family of Lemke graphs. Proving that we can doppelgang v_{k-2} and x seems straightforward, but the subdivision construction will most likely prove to be challenging.

8. Future work

Many questions arise from the new Lemke graphs. While we know the minimal Lemke graph has diameter three, and that none exist on diameter two [10], what is the minimal Lemke graph of diameter four? It might be B subdivided along the edges (a, x), (b, x), (c, x), and (d, x) with a path of length one. If this is true, then all Lemke graphs on 10 and 11 vertices are diameter three. So far we have determined that there are 5957 Lemke graphs of diameter three on ten vertices, including the one previously mentioned. Further, there are no 10-vertex Lemke graphs with diameter four.

For a given graph size, what values of q are possible for violators? We demonstrated a graph on 10 vertices that has violators with $q = 6$ and $q = 7$. It turns out that all 10-vertex Lemke graphs with diameter three have violators with $q = 6$ and/or $q = 7$. Do any 10-vertex Lemke graphs have other q values? In fact, are there any 10-vertex Lemke graphs with diameter larger than 4? As another example, for a graph $G \in S(1)$, the violating configurations have $q = 5$ and for a graph $H \in S(0)$, the violating configurations of $H(4, 0)$ have $q = 9$ (both graphs on 12 vertices). Are there graphs on 12 vertices that have violators with other q values, particularly $q \in \{6, 7, 8\}$?

All currently known violating configurations have $2\pi(G) - q + 1$ or $2\pi(G) - q + 2$ pebbles. Is it possible to find one with $2\pi(G) - q + 3$ or more pebbles, or to prove that it is impossible?

Gao and Yin have shown that Graham's conjecture holds for the Cartesian product of a graph in $A(t)$ with complete graphs and trees [6]. Can this result be expanded to any graph in $S(t)$?

Can anything be said about properties that all Lemke graphs have? All previously known Lemke graphs have the antipodal property (a graph has the *antipodal property* if given two vertices u, v where the distance between them is the diameter of the graph, for any other vertex y in the graph, there is a shortest path from u to v through y [10]). However with the addition of doppelgangers, we know of Lemke graphs without this property. None of the currently known Lemke graphs provide a counterexample to the conjecture that all bipartite graphs have the two-pebbling property [11].

The maximal and minimal Lemke graphs on 8 vertices could be crucial to the future of Graham's Conjecture. While $L\square L$ has been of interest, so far it has been too large to computationally analyze. The additional edges in $A\square A$ could aid in computationally calculating its pebbling number since so far in practice our algorithms have been faster on products involving A than L. On the other hand, if a counterexample to Graham's conjecture exists, $B\square B$ now seems to be the most likely candidate.

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