Ray Tracing Hit Times

Suppose $r(t) = S + \boldsymbol{c}t$ is a ray where S is a point of the form $S = \langle S_x, S_y, S_z\rangle$ and $\boldsymbol{c}$ is a vector of the form $\boldsymbol{c} = \langle c_x, c_y, c_z\rangle$. We'll calculate the hit times for the ray hitting a generic sphere, a generic plane, and a generic cylinder.

In general, a ray $r(t)$ will hit a shape $F(x, y, z) = 0$ when (and if) $F(r(t)) = 0$. So we'll be solving $F(r(t)) = 0$ for t in various contexts.

Generic Sphere The generic sphere is a sphere of radius 1 centered at the origin. Its form is $F(x, y, z) = 0$ where $F(x, y, z) = x^2 + y^2 + z^2 - 1$. To calculate the hit times, we solve $F(r(t)) = 0$ for t.

$$\begin{gathered}(S_x + c_xt)^2 + (S_y + c_yt)^2 + (S_z + c_zt)^2 - 1 = 0\\ S_x^2 + 2S_xc_xt + c_x^2t^2 + S_y^2 + 2S_yc_yt + c_y^2t^2 + S_z^2 + 2S_zc_zt + c_z^2t^2 - 1 = 0\\ (c_x^2 + c_y^2 + c_z^2)t^2 + 2(S_xc_x + S_yc_y + S_zc_z)t + (S_x^2 + S_y^2 + S_z^2 - 1) = 0\end{gathered}$$

Using the quadratic formula yields

$$t_{hit_1}, t_{hit_2} = \frac{-(S\cdot\boldsymbol{c}) \pm \sqrt{(S\cdot\boldsymbol{c})^2 - \|\boldsymbol{c}\|^2(\|S\|^2 - 1)}}{\|\boldsymbol{c}\|^2}$$

Generic Plane The generic plane is the xy-plane, $z = 0$. So, $F(x, y, z) = z$. Thus, $S_z + c_zt = 0$. So, $t_{hit} = -S_z/c_z$.

Generic Cylinder The generic cylinder is a cylinder of radius 1 and length 1 with a central axis coincident with the z-axis that sits on the xy-plane. That is, its side satisfies $x^2 + y^2 - 1 = 0$ with $0 < z < 1$ while its ends are disks of radius 1 contained in the planes $z = 0$ and $z = 1$.

To find hit times for the generic cylinder, we have to determine whether the ray hits the side of the cylinder or one of the ends. We'll start with the side. First see if the ray hits an infinitely long cylinder of radius 1 about the z-axis:

$$\begin{gathered}(S_x + c_xt)^2 + (S_y + c_yt)^2 - 1 = 0\\ (c_x^2 + c_y^2)t^2 + 2(S_xc_x + S_yc_y)t + (S_x^2 + S_y^2 - 1) = 0\end{gathered}$$

Using the quadratic equation again yields two *potential* hit times:

$$t_{hit_1}, t_{hit_2} = \frac{-(S_xc_x + S_yc_y) \pm \sqrt{(S_xc_x + S_yc_y)^2 - (c_x^2 + c_y^2)(S_x^2 + S_y^2 - 1)}}{c_x^2 + c_y^2}$$

Now to determine if the generic cylinder is actually hit, we need to check whether the z-coordinate of each $r(t_{hit})$ is between 0 and 1. If it is, then the ray hits the side of the generic cylinder. If not, then it doesn't, although it may still hit an end of the cylinder.

To check whether the ray hits an end, the ray must hit the appropriate plane, $z = 0$ or $z = 1$. If $S_z + c_z t_0 = 0$ for some t_0 (i.e., if $t_0 = -S_z/c_z$), then the ray passes through the xy-plane ($z = 0$) at that time. If, furthermore, the ray passes through the xy-plane within the circle of radius 1 centered at the origin, then the ray has hit the end of the cylinder. That is, if

$$\left(S_x - \frac{c_x S_z}{c_z}\right)^2 + \left(S_y - \frac{c_y S_z}{c_z}\right)^2 \leq 1,$$

then the ray hit the end of the generic cylinder coincident with the xy-plane at time t_0.

Similarly, the ray hits the plane $z = 1$ when $t_1 = (1 - S_z)/c_z$. So the other end of the cylinder will be hit at time t_1 if

$$\left(S_x + \frac{c_x(1 - S_z)}{c_z}\right)^2 + \left(S_y + \frac{c_y(1 - S_z)}{c_z}\right)^2 \leq 1.$$